Sketch of Solutions for Tutorial 5

Tutorial 5:

1. *Prove that every maximal matching in a graph G has size at least half the size of a maximum matching.*

   Consider a maximum matching M of size α'. Then, for every edge in M, at least one endpoint of the edge must be present in any maximal matching (or the edge can be added to the matching, and hence it is not maximal). Hence, the number of vertices covered by any maximal matching must at least be $|M| = \alpha'(G)$. Since an edge in the mathing can cover at most 2 vertices, the number of edges in the maximal matching must be $\alpha'(G)/2$.

2. *Prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$).*

   Each vertex in a vertex cover can cover at most $\Delta(G)$ edges. By definition of vertex cover, this implies that min vertex cover size $\geq$ $e(G)/\Delta(G)$. The result then follows from the Konig-Egervary theorem.

3. *Consider a bipartite graph G with partite sets X and Y. For any subset of vertices S, let N(S) denote the set of vertices such that there is an edge from some vertex in S to some vertex in N(S). Show that if $|N(S)| \geq |S|d$ for every subset S of X and some fixed positive integer d, then G has a matching of size $|X| - d$.*

   Add d new vertices to Y, and add an edge from every vertex of X to each of the D new vertices. Now, $|N(S)| \geq |S|$ for every subset S of X. Hence, by Hall's theorem, there exists a matching that saturates X. At most d edges of this matching can go to the d new vertices. Hence, there still remains a matching of size at least $|X| - d$. Therefore, there exists a matching of size $|X| - d$.

4. *Prove that if a graph G decomposes into 1-factors, then G has no cut vertex.*

   Since G has at least one 1-factor, G has even order. It can also be seen easily that G is k-regular (as it is a decomposition of 1-factors, each 1-factor taking one edge off a vertex). Assume $k > 1$ ($k = 1$ is just a collection of components each of which is a single edge only).

   Consider any vertex v in G. Since G has even order, $G - v$ has at least one component H of odd order. Then v must be matched to a vertex in H in any 1-factor of G, or all vertices in H cannot be matched. Also, for any vertex u in H, it must be matched only with vertices in H or v. Since there are k 1-factors involving each such $u \in H$, all k edges of v must be used to match vertices in H. Hence, $G - v$ cannot have any other component other than H (as then v must have at least one edge to some vertex in that component). Hence v is not a cut vertex. Since this is true for any $v \in G$, G does not have a cut vertex.