Sketch of Solutions for Tutorial 4 and Homework 4

Tutorial 4:

1. *Prove that a k-regular bipartite graph has no cut edges.*

   Proof by contradiction. Suppose that it has a cut edge. Then removal of that edge will divide the graph into two connected components C_1 and C_2. Each of these components is a bipartite graph itself. Taking any of these two components, say C_1, one partite set has all vertices with degree k, and the other partite set has one vertex of degree $k-1$ and all other vertices of degree k (since only one edge is removed). Counting the nummbr of edges incident on each partite set will then show a mismatch between the two partite sets, which is a contradiction.

2. *Let G be a k-connected graph, and let S, T be disjoint subsets of V(G) with size at least k. Prove that G has k pairwise disjoint S,T paths.*

   Add a new vertex x with edges to all vertices in S and a new vertex y with edges to all vertices in T. Then this new graph is k-connected (by Expansion Lemma). So there is at least k internally disjoint paths between x and y. Remove x and y from each of these paths to get the required paths.

3. *Use Mengers theorem to prove that $\kappa(G) = \kappa'(G)$ when G is 3-regular.*

   By Menger's theorem, for any two vertices x, y, there are $\kappa'(G)$ pairwise edge-disjoint paths between x and y. These paths must also be vertex disjoint, or some vertex will have at least 4 edges incident on it. So $\kappa(G) \geq \kappa'(G)$. But we know that $\kappa(G) \leq \kappa'(G)$. Henxe $\kappa(G) = \kappa'(G)$.

Homework 4:

1. *Let v be a vertex of a 2-connected graph G. Prove that v has a neighbor u such that $G - u - v$ is connected .*

   Since G is 2-connected, $G - v$ is connected. if $G - v$ is 2-connected, choose any neighbor u of v, and $G - u - v$ is still connected. If $G - v$ is not 2-connected, consider a maximal connected subgraph H of $G - v$ containing exactly one cut-vertex x of $G - v$ (see the notion of Blocks and their properties in West to verify that such an H will exist). Then, since $G - x$ is connected, v must have at least one neighbor in $H - x$, as otherwise $G - x$ is disonnected with H as one of the components. Let u be the neighbor of v in $H - x$. Since $H - u$ is connected, $G - u - v$ is connected.

2. *Prove that a simple connected graph with an even number of edges can be decomposed into paths with two edges (P_3).*

   Proof by induction on no. of edges e.

   Base case: $e = 2$. This is just P_3 itself.

   Induction Hypothesis: For all graphs G with even no. of edges $e < k$, G can be decomposed into P_3's.

   Consider a connected graph G with k edges, where k is even. Consider an arbitrary path $P =< i, j, k >$ of length 2. Consider $G' = G - P$. Then G' has even no. of edges (sice two edges are removed). Since P has 2 edges, G' has at most 3 components (deletion of one edge can increase the number of components by at most 1). If all the components have even no. of edges, we apply the induction hypothesis to each of them (as each has less than k edges) to get the required decomposition.

   Otherwise, note that exactly two components G_1 and G_2 can have odd no. of edges (as the total no. of edges is even). Also, G_1 and G_2 will both contain at least one of i, j, k such that G_1 and G_2 contain disjoint subsets of i, j, k (note that if they are not disjoint, G_1 and G_2 has a vertex in common, so they cannot be different components in G'). Without loss of generality, let G_1 contain i and G_2 contain j. Then, decompose the original graph G as follows: $G_3 = G_1$ plus the edge (i, j), $G_4 = G_2$ plus the edge (j, k), and any other third component G_5 that may have been there on removal of the path P (note that G_5, if present, has even number of edges). Now each of G_3, G_4, and G_5 (if present) has even no. of edges with no. of edges less than k. We apply the induction hypothesis to each of them to get the required decomposition.