Sketch of Solutions for Tutorial 3 and Homework 3

Tutorial 3:

1. *Show that a graph with n vertices with vertex connectivity k has at least kn/2 edges.* (3)

   Vertex-connectivity k means that $\delta(G) \geq k$. Hence total number of edges = (Sum of all degrees)/2 $\geq kn/2$.

2. *Prove that every edge cut in a graph must contain one branch from every spanning tree of the graph.* (3)

   Suppose not, i.e., an edge cut X does not contain any branch of some spanning tree. But then, G - X is still connected, as a spanning tree can be built in G. Hence X cannot be an edge cut. This is a contradiction.

3. *Prove that an Eulerian graph cannot have a minimal edge cut with an odd number of edges.* (4)

   Consider any minimal edge cut X. Then, G - X has exactly two components (or X will not be minimal). Since G is Eulerian, G can be decomposed into cycle. If a cycle lies entirely in one component, no edge of the cycle belongs to X. However, if a cycle lies partly in one component and partly in the other, there must be an even number of edges from that cycle in X (as if we traverse the cycle in any order starting from one component, for every edge going to the other component (and hence belongs to X), there must be an edge coming back (and hence must also belong to X).

   So X contains an even number of edges from each of the cycles that G can be decomposed into. Hence the result.

Homework 2:

1. *Prove that no graph has a cut vertex of degree 1.* (5)

   Consider a cut vertex v of G. Then $G-v$ has more than one component. Let (G - v) have components $C_1, C_2, \ldots, C_k$, $k > 1$. Consider any two vertices $x \in C_i$ and $y \in C_j$, $i \neq j$. Since G is connected, there is a path between x and y in G. Since C_i and C_j are in different components of $G - v$, there is no path between x and y in $G - v$. Hence the path between x and y in G must contain v (or the path would have remained on removal of v). Also, v is not an endpoint of this path (as $x, y \neq v$). Hence v must have at least two neighbors (the two nodes it is connected to in the path). Hence. $d(v) > 1$.

2. *Prove that $\kappa(G) = \kappa'(G)$ when G is a simple graph with $\Delta(G) \leq 3$.* (5)

   The case $\Delta(G) = 1$ is obvious. For $\Delta(G) = 2$, if $\delta(G) = 1$, κ(G) = κ'(G) = 1 (remove the edge or the node connecting the degree 1 vertex with its neighbor. If $\delta(G) = 2$, each component is a cycle (min and max degree = 2 means degree of all nodes = 2), and κ(G) = κ'(G) = 2.

   For $\Delta(G) = 3$, see proof of Theorem 4.1.11 in West. If $\delta(G) = 3$, it is a 3-regular graph, and the theorem proves it. If $\delta(G) = 2$, the proof is similar.

3. *Let v be a cut vertex of a simple connected graph G. Prove that $(G' - v)$ is connected.* (5)

   Since v is a cut vertex of G, let $(G-v)$ have components $C_1, C_2, \ldots, C_k$, $k > 1$.

   Consider any two vertices $x \in C_i$ and $y \in C_j$, $i \neq j$. Then $(x, y) \notin E(G)$. Therefore $(x, y) \in E(G')$. Thus any two vertices in two different components of $G - v$ are connected by a direct edge in $G' - v$.

   Now consider any two vertices (x, y) in the same component C_i. If $(x, y) \notin E(G)$ then $(x, y) \in E(G')$. If $(x, y) \in E(G)$, then $(x, y) \notin E(G')$. But then, consider any node $z \in C_j, i \neq j$. We have already shown that $(x, z) \in E(G' - v)$ and $(y, z) \in E(G' - v)$. Hence there is a path between x and y in $(G' - v)$.

   Thus, there is a path between any two vertices in $(G' - v)$. hence $(G' - v)$ is connected.

4. *Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an (x,z) path through y.* (5)

   Only if: Since G is 2-connected, x and y must lie on some cycle C1 and y and z must lie on some cycle C2, and C1 and C2 has at least y in common. Now just think of the different cases possible (Case 1: C1 and C2 has only y in common and Case 2: C1 and C2 has more than one vertex in common. The second case will have subcases).

   If: G is conected as there is a path between any two vertiecs. Suppose there is a cut-vertex v. Consider any two vertices x and y in two different components in $G - \{v\}$. Then, there exists a path from x to v through y. But this path cannot go through v. So there exists a path from x to y not including v. So they must be in the same component. Hence v cannot be a cut vertex. So G is 2-connected.

5. *Let G be a connected graph with at least 3 vertices. Form G' from G by adding an edge (x,y) in G' (if it is not already there in G) whenever d(x,y) = 2 in G. Prove that G' is 2-connected.* (5)

   Suppose not. Then there exists a cut vertex v in G'. Since v is a cut vertex, $\exists x, y \in V$ such that x, y are neighbors of v, (x, y) is not an edge in G', and x, y belongs to different components of G - v. This means $d(x, y) > 2$ in G. Since no edges of G are removed, this means that at least one or both of x or y is at distace 2 from v in G (or they will not be at distance 2 in G'). Without any loss of generality, let that node be x. Then, $\exists z \in V$ such that z is a neighbor of both x and v. But then, $d(z, y) = 2$ (if $(y, v) \in E$, or $d(z, p) = 2$ (if $d(y, v) = 2$ in G and p is the intermediate node in that path). But then, there must be an edge from z to y (or p) in G'. Note that this is an edge between two direct neighbors of v in two different compoenents, and this will be true for any such neighbors. Hence v cannot be a cut vertex. This is a contradiction.