Sketch of Solutions for Tutorial 2 and Homework 2

Tutorial 2:

1. *Let T be a tree of order n. Prove that T is isomorphic to a subgraph of C'_{n+2} (complement of C_{n+2}).* (3)

   The minimum degree of a node in C'_{n+2} is $n-1$. The oder of the tree is nn, so there are $n-1$ edges. Now use the known result that a tree with m edges is a subgraph of a graph with min. degree $\geq m$.

2. *The girth of a graph G, denoted by $g(G)$, is the length (no. of edges) of the smallest cycle in the graph. Prove that a k-regular graph with girth 4 has at least $2k$ vertices. What is the class of such graphs (i.e. k-regular with girth 4) with exactly $2k$ vertices?* (3)

   Consider any edge $(u, v) \in G$. Since G is k-regular, degree of both u and v is k. let the neighbors of u be $u_1, u_2, \ldots, u_k$ and the neighbors of v be $v_i, v_2, \ldots, v_k$. Consider any neighbor u_i of u, $u_i \neq v$, and any neighbor v_j of v, $v_j \neq u$. Then, $u_i \neq v_j$ for any i, j, as then we will have a cycle of length 3 (with the edges $(u, u_i), (u_i, v)$, and (v, u)). Hence the number of nodes in G is at least $(k-1)+(k-1)+2 = k$ ($k-1$ neighbors of u excluding v, $k-1$ neighbors of v excluding u, plus u and v).

   The required class is $K_{k,k}$, i.e., complete bipartite graph with k vertices in each partite set.

3. *Prove that for every 3-regular graph, the vertex connectivity is the same as the edge connectivity.* (3)

   See Theorem 4.1.11 in West.

Homework 2:

1. *Prove that if W is a nontrivial closed walk that does not contain a cycle, then some edge of W occurs twice in succession (once in each direction).* (5)

   Since W is a closed walk with no cycle, there must be at least one repeated vertex ω other than the starting vertex (or there is a cycle). Consider two consecutive repeations of ω. Then, there is a closed walk $W' = (\omega, v_1, \ldots, v_k, \omega)$ in W. Consider the ω, v_k-walk in W'. Then, there exists a ω, v_k-path in it. If the length of this path is greater than 1, then this path plus the edge (v_k, ω) will form a cycle in W', and hence in W. This is a contradiction. Hence, the ω, v_k-path must be of length 1, meaning it is a single edge. Therefore, $v_1 = v_k$ in W'. Hence the edge (ω, v_k) repeats twice in succession.

2. *Prove that G has a Hamiltonian Path only if for every $S \subseteq V$, the number of components of $G - S$ is at most $|S| + 1$.* (5)

   Suppose the components of G - S are $C_1, C_2, \ldots, C_k$. Since a hamiltonian path in G must cover all vertices, it must go to each of the components of $G - S$. To go from one component to another, there must be one vertex in S traversed in between (as there is no direct path between two components of $G - S$). Hence the maximum number of components can only be $|S| + 1$ (happens when the two endpoints of the ham path are outside of S).

3. *The girth of a graph G, denoted by g(G), is the length (no. of edges) of the smallest cycle in the graph. Prove that every graph G containing at least one cycle satisfies the relation g(G) $\leq$ 2diam(G) + 1, where diam(G) is the diameter of G.* (5)

   Proof by contradiction. Let C be a shortest cycle in G and let $g(G) \geq 2diam(G) + 2$. Then there exists two vertices x and y on C such that the distance between them in C is at least $diam(G) + 1$. By definition of diameter, $d(x, y)$ in G is less than $diam(G) + 1$, so a shortest path P between x and y in G is not a subgraph of C. Consider the last intersection of P with some vertex of C before y. Let this vertex be w. Then consider the cycle formed by the part of P from w to y (this will be of length $\leq diam(G)$), and the shorter of the two paths between w and y in C (which will not have any vertex common with the part of P

from w to y, and is of length $\leq diam(G) + 1$). This is a shorter cycle than C, which is a contradiction.

4. *If G is a graph, then the line graph of G, L(G), is the graph formed as follows: For each edge in G, add a vertex in L(G) and add an edge between two vertices in L(G) if the corresponding edges in G have a common endpoint. Prove that*

   (a) *If G is Eulerian, then L(G) has a Hamiltonian circuit*

   (b) *$L(K_{m,n})$ is regular*

   (3 + 3)

   Part 1: Consider an Euler tour P of G. Hence every edge of G occurs exactly once in P. Consider the sequence S of vertices in L(G) corresponding to the edges in P. Then, every vertex of L(G) occurs exactly once in S. Also, for any two consecutive vertices u, v in S, the corresponding edges in G must have a common endpoint (or the Euler tour could not have them in sequence. Hence $(u, v) \in E(L(G))$. Hence S is a Hamiltonian cycle.

   Part 2: Consider a node in $L(K_{m,n})$. The edge corresponding to it in $K_{m,n}$ has $(n - 1)$ adjacent edges in one partition and $(m - 1)$ adjacent edges in the other partition. Since this is true for any node in $L(K_{m,n})$, hence it is (m+n-2)-regular.

5. *Let v be a vertex in a connected graph G. Prove that there exists a spanning tree T of G such that the distance of every vertex from v is the same in G and in T.* (5)

   Construct the shortest paths from v to all other vertices in G, with the following constraint: if the shortest path from v to some u contains w, then the shortest path from v to w chosen is the subpath of the path to u (this is to prevent the following scenarios which can cause a cycle: suppose there are two shortest paths from v to w, namely P_1 and P_2. If the path from v to u includes P_1, and P_2 is chosent as the path from v to w, then there is a cycle containing v, w, $P - 1$, and P_2). Consider the graph $G_1 = (V, E_1)$ where E_1 is the union of the edges in all the shortest paths.

That G_1 is connected is easy to see (for any x, y, there is at least the path from x to v and v to y if the paths don't share any vertex other than v. If there are any common vertices, branch off from there).

We next show that G_1 is acyclic. Suppose there is a cycle C in G_1. Consider any node $x \in C$. Then there are two paths from v to x in C. But only one path is added. This is a contradiction. So G_1 is acyclic. (this can be argued simply from the optimal substructure property of shortest paths, given the way we have chosen the shortest paths).

Hence G_1 is a spanning tree. That the distances from v to all other vertices are the same in G and T is obvious from the construction of T.