# Sketch of Solutions for Tutorial 1 and Homework 1

**Tutorial 1:**

1. *Prove that every n-vertex graph with n edges contains a cycle* (3)

   Consider $n$ vertices and add the edges one by one. If an edge is added between two nodes of the same component, we have a cycle and the proof is done. If not, the addition of the edge will reduce the number of components by 1. Hence, after adding $(n-1)$ edges, if no cycle is still formed, there is exactly one component. Then the $n^{th}$ edge must create a cycle.

2. *Consider a k-regular bipartite graph with m and n vertices in the two bipartite sets. Prove that $m = n$.* (3)

   Count the number of edges incident on each partite set. The no. of edges incident on the partite set with $m$ vertices is $k.m$. Similarly the no. of edges incident on the partite set with $n$ vertices is $k.n$. Since these are all the edges, they must be the same. Hence $m = n$.

3. *Suppose that a connected graph $G$ is decomposed into two graphs $G_1$ and $G_2$. prove that $G_1$ and $G_2$ must have a common vertex.* (3)

   Given:
   G is a connected, simple graph with $V(G_1) \cup V(G_2) = V(G)$, $E(G_1) \cup E(G_2) = E(G)$, and $E(G_1) \cap E(G_2) = \phi$ (since $G_1, G_2$ is a decomposition).

   To show that $V(G_1) \cap V(G_2) \neq \phi$.

   We prove by contradiction. Suppose $V(G_1) \cap V(G_2) = \phi$.
   $\Rightarrow$ there does not exist $u, v$ such that $u \in V(G_1)$, $v \in V(G_2)$ and

$(u, v) \in E(G)$ (otherwise the edge would have been part of one of $G_1$ or $G_2$, and its one end point would be common to both.
$\Rightarrow \exists x, y,\ x \in V(G_1),\ y \in V(G_2)$ such that there is no path between $x$ and $y$
$\Rightarrow G$ is not connected. This is a contradiction.

4. *Let $P$ and $Q$ be two paths of maximum length in a connected graph $G$. Prove that $P$ and $Q$ must have a common vertex.* (3)

   Proof is again by contradiction. Suppose there is no such common vertex. Since $G$ is connected, $\exists u, v,\ u \in P,\ v \in Q$ such that there is a path from $u$ to $v$ **that does not include any other vertices in $P$ and $Q$** (Just as an aside, note that $u$ and $v$ cannot both be the endpoints of $P$ and $Q$ as then $P$ and $Q$ can be merged to get a longer path).

   Let $u$ divide $P$ into paths $P_1$ and $P_2$ with lengths $x_1$ and $x_2$ respectively. Without loss of generality, assume that $x_1 \geq x_2$. Similarly, let $v$ divide $Q$ into paths $Q_1$ and $Q_2$ with lengths $y_1$ and $y_2$ respectively. Without loss of generality, assume that $y_1 \geq y_2$. Then, $x_1 \geq M/2$ and$y_1 \geq M/2$ where $M = x_1 + x_2 = y_1 + y_2$.

   Consider the path $P'$ formed by $< P_1 >, u, v, < Q_1 >$. Then $|P'| > x_1 + y_1$. Hence $|P'| > M$. Hence $P$ and $Q$ could not have been the maximum length paths in $G$. This is a contradiction.

**Homework 1:**

1. *Checking for Isomorphism* $(2 \times 3 = 6)$

   Pr. 1.1.16: Not Isomorphic (complement of one is connected while the other is disconnected)
   Pr. 1.1.18: The first two are isomorphic (what is the mapping?), the third is not (has an odd cycle, the first two does not
   Pr. 1.1.19: The second two are isomorphic (mapping?), the first is not (the first is bipartite, the last two contain odd cycles)

2. *Checking for graphic sequence* $(1.5 \times 4 = 6)$

   Pr 1.3.8: The first three are graphic, the last is not. Apply Havel-Hakimi directly to check.

3. *Prove or disprove: If G is an Eulerian graph, and there are two edges e and f in G sharing a vertex, then G has an Eulerian circuit in which e and f appear consecutively.* (3)

   False. Consider two edge-disjoint cycles touching at a node, and consider two edges of the same cycle incident at that node. They cannot occur consecutively.

4. *Let G be a graph with at least 2 vertices. Prove or disprove the following:*

   *(a) Deleting a vertex with degree ?(G) cannot increase the average degree of G.*

   *(b) Deleting a vertex with degree ?(G) cannot reduce the average degree of G.*

   $(3 \times 2 = 6)$

   For part 1, True. Write the average degree before and after the deletion, the math is simple from there to show it cannot increase. just remember that if a vertex with degree $d$ is deleted, the total degree of the graph decreases by $2d$.

   For part 2, False. Consider $K_n$ and remove any vertex (all vertices have same degree, so $\delta(K_n) = n - 1$. Average degree decreases from $(n - 1)$ to $(n - 2)$.

5. *Prove that every n-vertex graph with m-edges has at least m-n+1 cycles.* (3)

   Consider a spanning forest of G. This has maximum (n-1) edges (as there are no cycles). Adding each of the remaining edges creates exactly one cycle (as it is added to some spanning tree of a component). Also all these cycles are distinct (as at least one edge is different). Hence the minimum no. of cycles $= m - (n - 1) = m - n + 1$.

6. *Let G be a connected simple graph not having $P_4$ or $C_3$ as an induced sub-graph. Prove that, G is a complete bipartite graph (where $P_n$ is a path with n vertices and $C_n$ is a cycle with n vertices).* (7)

   Consider a vertex $x$ and the set of its neighbors $N(x)$. For all $u, v \in N(x)$, there can be no edge $(u, v)$ in the graph since there is no $C_3$. All other vertices $V(G) - x - N(x)$ must have a path to $x$ since the graph is connected. This path contains a vertex from $N(x)$. Each vertex $u \in V(G) - x - N(x)$ has edges with all the vertices in $N(x)$ otherwise there would be an induced $P_4$. For all $u, v \in V(G) - x - N(x)$, there can be no edge $(u, v) \in E(G)$ since there is no $C_3$. Thus, $x \cup \{V(G) - x - N(x)\}$ forms one partite set while $N(x)$ forms the other partite set. It is easy to see that it is complete.