Sketch of Solutions for Tutorial 1 and Homework 1

Tutorial 1:

1. Prove that every n-vertex graph with n edges contains a cycle. (3)
   Consider n vertices and add the edges one by one. If an edge is added between two nodes of the same component, we have a cycle and the proof is done. If not, the addition of the edge will reduce the number of components by 1. Hence, after adding \((n - 1)\) edges, if no cycle is still formed, there is exactly one component. Then the \(n^{th}\) edge must create a cycle.

2. Consider a k-regular bipartite graph with \(m\) and \(n\) vertices in the two bipartite sets. Prove that \(m = n\). (3)
   Count the number of edges incident on each partite set. The no. of edges incident on the partite set with \(m\) vertices is \(k.m\). Similarly the no. of edges incident on the partite set with \(n\) vertices is \(k.n\). Since these are all the edges, they must be the same. Hence \(m = n\).

3. Suppose that a connected graph \(G\) is decomposed into two graphs \(G_1\) and \(G_2\). prove that \(G_1\) and \(G_2\) must have a common vertex. (3)
   Given:
   \(G\) is a connected, simple graph with \(V(G_1) \cup V(G_2) = V(G), E(G_1) \cup E(G_2) = E(G)\), and \(E(G_1) \cap E(G_2) = \emptyset\) (since \(G_1, G_2\) is a decomposition).
   
   To show that \(V(G_1) \cap V(G_2) \neq \emptyset\).
   
   We prove by contradiction. Suppose \(V(G_1) \cap V(G_2) = \emptyset\).
   
   \(\Rightarrow\) there does not exist \(u, v\) such that \(u \in V(G_1), v \in V(G_2)\) and
\((u, v) \in E(G)\) (otherwise the edge would have been part of one of \(G_1\) or \(G_2\), and its one end point would be common to both.

\[ \Rightarrow \exists x, y, x \in V(G_1), y \in V(G_2) \text{ such that there is no path between } x \text{ and } y \]

\[ \Rightarrow G \text{ is not connected. This is a contradiction.} \]

4. Let \(P\) and \(Q\) be two paths of maximum length in a connected graph \(G\). Prove that \(P\) and \(Q\) must have a common vertex.

Proof is again by contradiction. Suppose there is no such common vertex. Since \(G\) is connected, \(\exists u, v, u \in P, v \in Q\) such that there is a path from \(u\) to \(v\) that does not include any other vertices in \(P\) and \(Q\) (Just as an aside, note that \(u\) and \(v\) cannot both be the endpoints of \(P\) and \(Q\) as then \(P\) and \(Q\) can be merged to get a longer path).

Let \(u\) divide \(P\) into paths \(P_1\) and \(P_2\) with lengths \(x_1\) and \(x_2\) respectively. Without loss of generality, assume that \(x_1 \geq x_2\). Similarly, let \(v\) divide \(Q\) into paths \(Q_1\) and \(Q_2\) with lengths \(y_1\) and \(y_2\) respectively. Without loss of generality, assume that \(y_1 \geq y_2\). Then, \(x_1 \geq M/2\) and \(y_1 \geq M/2\) where \(M = x_1 + x_2 = y_1 + y_2\).

Consider the path \(P'\) formed by \(< P_1 >, u, v, < Q_1 >\). Then \(|P'| > x_1 + y_1\). Hence \(|P'| > M\). Hence \(P\) and \(Q\) could not have been the maximum length paths in \(G\). This is a contradiction.
Homework 1:

1. **Checking for Isomorphism**
   
   Pr. 1.1.16: Not Isomorphic (complement of one is connected while the other is disconnected)
   
   Pr. 1.1.18: The first two are isomorphic (what is the mapping?), the third is not (has an odd cycle, the first two does not
   
   Pr. 1.1.19: The second two are isomorphic (mapping?), the first is not (the first is bipartite, the last two contain odd cycles)

2. **Checking for graphic sequence**

   Pr 1.3.8: The first three are graphic, the last is not. Apply Havel-Hakimi directly to check.

3. **Prove or disprove: If** $G$ **is an Eulerian graph, and there are two edges** $e$ **and** $f$ **in** $G$ **sharing a vertex, then** $G$ **has an Eulerian circuit in which** $e$ **and** $f$ **appear consecutively.**
   
   False. Consider two edge-disjoint cycles touching at a node, and consider two edges of the same cycle incident at that node. They cannot occur consecutively.

4. **Let** $G$ **be a graph with at least 2 vertices. Prove or disprove the following:**
   
   (a) Deleting a vertex with degree $\delta(G)$ cannot increase the average degree of $G$.
   
   (b) Deleting a vertex with degree $\delta(G)$ cannot reduce the average degree of $G$.

   For part 1, True. Write the average degree before and after the deletion, the math is simple from there to show it cannot increase. just remember that if a vertex with degree $d$ is deleted, the total degree of the graph decreases by $2d$.

   For part 2, False. Consider $K_n$ and remove any vertex (all vertices have same degree, so $\delta(K_n) = n - 1$. Average degree decreases from $(n - 1)$ to $(n - 2)$.
5. Prove that every $n$-vertex graph with $m$-edges has at least $m-n+1$ cycles.

Consider a spanning forest of $G$. This has maximum $(n-1)$ edges (as there are no cycles). Adding each of the remaining edges creates exactly one cycle (as it is added to some spanning tree of a component). Also all these cycles are distinct (as at least one edge is different). Hence the minimum no. of cycles $= m - (n - 1) = m - n + 1$.

6. Let $G$ be a connected simple graph not having $P_4$ or $C_3$ as an induced sub-graph. Prove that, $G$ is a complete bipartite graph (where $P_n$ is a path with $n$ vertices and $C_n$ is a cycle with $n$ vertices).

Consider a vertex $x$ and the set of its neighbors $N(x)$. For all $u, v \in N(x)$, there can be no edge $(u, v)$ in the graph since there is no $C_3$. All other vertices $V(G) - x - N(x)$ must have a path to $x$ since the graph is connected. This path contains a vertex from $N(x)$. Each vertex $u \in V(G) - x - N(x)$ has edges with all the vertices in $N(x)$ otherwise there would be an induced $P_4$. For all $u, v \in V(G) - x - N(x)$, there can be no edge $(u, v) \in E(G)$ since there is no $C_3$. Thus, $x \cup \{V(G) - x - N(x)\}$ forms one partite set while $N(x)$ forms the other partite set. It is easy to see that it is complete.