Sketch of Solutions for Homework 3

1. Prove by induction that the maximum number of edges in an n-vertex, simple non-Hamiltonian graph is \((n-1)(n-2)/2 + 1\). 

Use induction on \(n\). Form the base cases for \(n=2,3\) (\(K_2, K_3\)). Consider true for graphs with \(n-1\) vertices. Now consider a graph \(G\) with \(n\) vertices and \(> (n-1)(n-2)/2 + 1\) edges. Then there will exist a vertex \(x\) with degree \(\geq n-2\) (show this easily). If you remove it from \(G\), you will see that the number of edges in \(G-x\) satisfies the induction hypothesis for \((n-1)\)-node graph, so \(G-x\) will contain a Ham-cycle. Now since \(x\) is non-adjacent to at most one vertex in \(G\), it must be adjacent to at least 2 consecutive vertices in that cycle. So the cycle can be extended to include \(x\) and create a Ham-cycle in \(G\).

2. If \(G\) is a graph, then the line graph of \(G\), \(L(G)\), is the graph formed as follows: For each edge in \(G\), add a vertex in \(L(G)\) and add an edge between two vertices in \(L(G)\) if the corresponding edges in \(G\) have a common endpoint. Prove that

   (a) If \(G\) is Eulerian, then \(L(G)\) has a Hamiltonian circuit
   (b) \(L(K_{m,n})\) is regular

   \((5 + 5)\)

**Part 1:** Consider an Euler tour \(P\) of \(G\). Hence every edge of \(G\) occurs exactly once in \(P\). Consider the sequence \(S\) of vertices in \(L(G)\) corresponding to the edges in \(P\). Then, every vertex of \(L(G)\) occurs exactly once in \(S\). Also, for any two consecutive vertices \(u, v\) in \(S\), the corresponding edges in \(G\) must have a common endpoint (or the Euler tour could not have them in sequence. Hence \((u, v) \in E(L(G))\). Hence \(S\) is a Hamiltonian cycle.
Part 2: Consider a node in $L(K_{m,n})$. The edge corresponding to it in $K_{m,n}$ has $(n-1)$ adjacent edges in one partition and $(m-1)$ adjacent edges in the other partition. Since this is true for any node in $L(K_{m,n})$, hence it is $(m+n-2)$-regular.

3. Prove that no graph has a cut vertex of degree 1. (5)

Consider a cut vertex $v$ of $G$. Then $G-v$ has more than one component. Let $(G - v)$ have components $C_1, C_2, \ldots, C_k, k > 1$. Consider any two vertices $x \in C_i$ and $y \in C_j, i \neq j$. Since $G$ is connected, there is a path between $x$ and $y$ in $G$. Since $C_i$ and $C_j$ are in different components of $G-v$, there is no path between $x$ and $y$ in $G-v$. Hence the path between $x$ and $y$ in $G$ must contain $v$ (or the path would have remained on removal of $v$). Also, $v$ is not an endpoint of this path (as $x, y \neq v$). Hence $v$ must have at least two neighbors (the two nodes it is connected to in the path). Hence. $d(v) > 1$.

4. Prove that $\kappa(G) = \kappa'(G)$ when $G$ is a simple graph with $\Delta(G) \leq 3$. (5)

The case $\Delta(G) = 1$ is obvious. For $\Delta(G) = 2$, if $\delta(G) = 1$, $\kappa(G) = \kappa'(G) = 1$ (remove the edge or the node connecting the degree 1 vertex with its neighbor. If $\delta(G) = 2$, each component is a cycle (min and max degree = 2 means degree of all nodes = 2), and $\kappa(G) = \kappa'(G) = 2$.

For $\Delta(G) = 3$, see proof of Theorem 4.1.11 in West. If $\delta(G) = 3$, it is a 3-regular graph, and the theorem proves it. If $\delta(G) = 2$, the proof is similar.

5. Is every connected $k$-regular graph ($k \neq 3$) nonseparable (i.e., has no cut vertex)? (5)

No. Easy to form a counterexample.