Planarity
A graph is planar if it can be drawn on a plane without crossings.

A plane graph is a particular drawing of a planar graph in the plane with no crossings.
Faces

- The *faces* of a plane graph are the maximal regions of the plane that are disjoint from the drawing.

- The length of a face in a plane graph $G$ is the length of the walk in $G$ that bounds it.
Dual Graphs

- Suppose G is a plane graph. The dual graph $G^*$ of G is a plane graph having a vertex for each face in G. The edges of $G^*$ correspond to the edges of G as follows:
  - If $e$ is an edge of G that has face X on one side and face Y on the other side, then the corresponding dual edge $e^* \in E(G^*)$ is an edge joining the vertices $x,y$ of $G^*$ that correspond to the faces X,Y of G that cuts e exactly once.
Results

- If $L(F_i)$ denotes the length of face $F_i$ in a plane graph $G$, then $2e(G) = \sum L(F_i)$.

- The following statements are equivalent for a plane graph $G$:
  - $G$ is bipartite.
  - Every face of $G$ has even length.
  - The dual graph $G^*$ is Eulerian.
Euler’s Formula & other results

- [Euler’s Formula:] If a connected plane graph \( G \) has \( n \) vertices, \( e \) edges and \( f \) faces, then
  \[ n - e + f = 2 \]

- If \( G \) is a simple planar graph with at least three vertices, then \( e(G) \leq 3n(G) - 6 \).
  - If \( G \) is also triangle-free, then \( e(G) \leq 2n(G) - 4 \)
K₅ and K₃,₃

- Two famous Kuratowski graphs
- Claim 1: K₅ is non-planar
  - Since no. of edges = 10 > (3(5) – 6 = 9)
- Claim 2: K₃,₃ is non-planar
  - K₃,₃ has no odd cycle, so all faces in a planar embedding of it has length at least 4
  - No. of edges = 9 > (2(6) – 4 = 8)
Kuratowski’s Theorem

- **Subdividing** an edge means replacing the edge with a path of length 2.
- **[Kuratowski’s Theorem]**: \( G \) is planar if and only if \( G \) contains no sub-division of \( K_5 \) or \( K_{3,3} \).
Coloring Planar Graphs

- [4-color Theorem]: Every planar graph is 4-colorable.
  - Long and hard proof
- Every planar graph can be colored with 5 colors
  - Easy constructive proof that gives a 5-coloring of any planar graph.