Planarity

Planar Graph

- A graph is *planar* if it can be drawn on a plane without crossings.
- A *plane graph* is a particular drawing of a planar graph in the plane with no crossings.

Faces

- The *faces* of a plane graph are the maximal regions of the plane that are disjoint from the drawing.
- The length of a face in a plane graph G is the length of the walk in G that bounds it.

Dual Graphs

- Suppose G is a plane graph. The *dual graph G** of G is a plane graph having a vertex for each face in G. The edges of G* correspond to the edges of G as follows:
  - If *e* is an edge of G that has face X on one side and face Y on the other side, then the corresponding dual edge $e^* \in E(G^*)$ is an edge joining the vertices *x,y* of G* that correspond to the faces X,Y of G that cuts *e* exactly once.

Results

- If $L(F_i)$ denotes the length of face F_i in a plane graph G, then $2e(G) = \Sigma L(F_i)$.

- The following statements are equivalent for a plane graph G:
  - G is bipartite.
  - Every face of G has even length.
  - The dual graph G* is Eulerian.

Euler's Formula & other results

- [Euler's Formula:] If a connected plane graph G has *n* vertices, *e* edges and *f* faces, then

$$n - e + f = 2$$

- If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$.
  - If G is also triangle-free, then $e(G) \leq 2n(G) - 4$

K_5 and $K_{3,3}$

- Two famous *Kuratowski* graphs
- Claim 1: K_5 is non-planar
  - Since no. of edges = 10 > (3(5) – 6 = 9)
- Claim 2: $K_{3,3}$ is non-planar
  - $K_{3,3}$ has no odd cycle, so all faces in a planar embedding of it has length at least 4
  - No. of edges = 9 > (2(6) – 4 = 8)

Kuratowski’s Theorem

- *Subdividing* an edge means replacing the edge with a path of length 2.
- [Kuratowski’s Theorem]: G is planar if and only if G contains no sub-division of K_5 or $K_{3,3}$.

Coloring Planar Graphs

- [4-color Theorem]: Every planar graph is 4-colorable.
  - Long and hard proof
- Every planar graph can be colored with 5 colors
  - Easy constructive proof that gives a 5-coloring of any planar graph.