Paths and Cycles

Path & Cycle

- A *path* in a graph is a single vertex or an ordered list of *distinct* vertices $v_1, \ldots, v_k$ such that $v_{i-1}v_i$ is an edge for all $2 \leq i \leq k$.
  - the ordered list is a *cycle* if $v_k v_1$ is also an edge
  - A path is an *u,v-path* if *u* and *v* are respectively the first and last vertices on the path
  - A path of *n* vertices is denoted by P_n, and a cycle of *n* vertices is denoted by C_n.

Walk & Trail

- A *walk* of length k is a sequence, $v_0, v_1, \ldots, v_k$ of vertices and edges such that (v_{i-1}, v_i) *is an edge* for $1 \leq i \leq k$.
- A *trail* is a walk with no repeated edge.
  - A *path* is a walk with no repeated vertex
  - A walk is *closed* if it has length at least one and its endpoints are equal
  - A *cycle* is a closed trail in which “first = last” is the only vertex repetition
  - A *loop* is a cycle of length one

- Lemma: Every *u,v*- walk contains a *u,v*-path

Connected Graph

- A graph G is *connected* if it has a *u,v-path* for each pair $u,v \in V(G)$, otherwise it is *disconnected*
- A *component* of G is a maximal connected subgraph of G
  - *Trivial component* – component with no edges
  - Otherwise, *non-trivial component*

- Lemma: Every graph with *n* vertices and *k* edges has at least $n - k$ components
- Lemma: If a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.
- A *cut-edge* or *cut-vertex* of a graph is an edge or vertex whose deletion increases the number of components

Bipartite Graphs

- A graph G is *bipartite* if V(G) is the union of two disjoint sets such that each edge of G consists of one vertex from each set.
  - A complete bipartite graph is a bipartite graph whose edge set consists of all pairs having a vertex from each of the two disjoint sets of vertices
  - A complete bipartite graph with partite sets of sizes *r* and *s* is denoted by $K_{r,s}$

Theorem (Konig, 1936): A graph G is bipartite if and only if it does not contain any cycle of odd length.

Euler Graphs

- A graph is *Eulerian* if it contains a closed trail containing all edges
  - A closed trail in cyclic order is also called a *circuit*
  - *Eulerian circuit* or *Eulerian trail* – circuit or trail in graph containing all edges
- A graph is *even* if all its vertex degrees are even

Theorem: A finite graph G is *Eulerian* if and only if all its vertex degrees are even and all its edges belong to a single component.

Fleury’s Algorithm

Input: A graph G with one non-trivial component and at most two odd vertices.

(If the graph is disconnected, we will ignore the trivial components)

Initialization: Start at a vertex that has odd degree unless G is even, in which case start at any vertex.

Iteration: From the current vertex, traverse any remaining edge whose deletion from the graph does not leave a graph with two components. Stop when all edges have been traversed.

The Chinese Postman Problem

- Suppose a mail carrier traverses all edges in a road network, starting and ending at the same vertex.
  - The edges have non-negative weights representing distance or time.
  - We seek a closed walk of minimum total length that uses all the edges.

Hamiltonian Paths and Cycles

- Hamiltonian path is a path containing all vertices of the graph
- Hamiltonian cycle is a closed walk that traverses every vertex exactly once (except the starting vertex where the walk terminates)

- Lemma: In a complete graph with n vertices, if n is an odd number ≥ 3, then there are (n – 1)/2 edge disjoint Hamiltonian cycles
- Theorem (Dirac, 1952): A sufficient condition for a simple graph G to have a Hamiltonian cycle is that the degree of every vertex of G be at least n/2, where n = no. of vertices in G (≥ 3)
- Lemma (Ore, 1960): If d(u) + d(v) ≥ n for every pair of non-adjacent vertices u and v of a simple graph G, then G is Hamiltonian.

- Hamiltonian Closure of G: Graph obtained from G by iteratively adding edges between non-adjacent vertices u and v with $d(u)+d(v) \geq n$
- Theorem (Bondy-Chvatal, 1976): A simple graph G is Hamiltonian if and only if its closure is Hamiltonian.

Travelling Salesman Problem

- A set of cities connected by roads
- A salesman starts from a city and has to travel to each city exactly once, returning back to the starting city
- In what order should he travel so as to minimize the total distance travelled?