Graph Theory

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Course Information

- Course webpage
  - *cse.iitkgp.ac.in/~agupta/graph*
- Textbooks
  - Introduction to Graph Theory by Douglas West
- Other materials may be given by me as needed

Introduction

Graph

- A graph G = (V,E) with *n* vertices and *m* edges consists of:
  - a *vertex set* V(G) = $\{v_1, \ldots, v_n\}$, and
  - an *edge set* E(G) = $\{e_1, \ldots, e_m\}$, where each edge consists of two (possibly equal) vertices called its *endpoints.*
- We write *uv* for an edge *e={u,v}*, and say that *u* and *v* are adjacent (neighbors)
- A *simple graph* is a graph having no loops (self-loops) or multiple (parallel) edges

Digraph

- A *directed graph* or *digraph* G consists of a vertex set V(G) and an edge set E(G), where each edge is an ordered pair of vertices.
  - A *simple digraph* is a digraph in which each ordered pair of vertices occurs at most once as an edge.
  - Throughout this course we shall consider undirected simple graphs, unless mentioned otherwise.

Graphs as Relations

- Edges of a graph induce an adjacency relation from V to V. For simple undirected graphs, the relation is
  - symmetric
  - not reflexive.

Complement

- The *complement* G' of a simple graph G is the simple graph with vertex set V(G) and edge set defined by:
  - $uv \in E(G')$ if and only if $uv \notin E(G)$

Subgraph

- A *subgraph* of a graph G is a graph H such that:
  - $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
- An *induced subgraph* of G is a subgraph H of G such that E(H) consists of all edges of G whose endpoints belong to V(H)

Complete Graph / Clique

- A *complete graph* or a *clique* is a simple graph in which every pair of vertices is an edge.
  - We use the notation K_n to denote a clique of *n* vertices
  - The complement K_n' of K_n has no edges

Degree and Degree Sequence

- *Degree* of a vertex v in G, d(v) or $d_G(v)$
  - Number of edges incident to v
- *Regular* graph
  - All nodes have same degree
  - *k-regular :* all nodes have degree k
- Claim : The number of vertices with odd degree is always even.

Degree Sequence & Graphic Sequence

- Degree sequence of a graph G
  - List of vertex degrees
  - Usually written in non-increasing order
    - $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$
- A *graphic sequence* is a list of non-negative numbers that is the degree sequence of some simple graph.
  - A simple graph with degree sequence *d realizes d.*

Graphic: Necessary & Sufficient Condition

- For n>1, the non-negative integer list *d* of size *n* is graphic if and only if d' is graphic, where d' is the list of size $n - 1$ obtained from *d* by deleting its largest element Δ, and subtracting 1 from its Δ next largest elements.

  [Havel 1955, Hakimi 1962]

Graph Isomorphism

- An *isomorphism* from G to H is a bijection $f: V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.
  - We say that *G is isomorphic to H*, written as $G \equiv H$, if there is an isomorphism from G to H.
  - Is isomorphism an equivalence relation?