Coloring

K-coloring

- A *k-coloring* of G is a labeling $f:V(G) \rightarrow \{1,\ldots,k\}$.
  - The labels are *colors*
  - The vertices with color *i* are a *color class*
  - A *k*-coloring is *proper* if $f(x) \neq f(y)$ for all (x, y) in $E(G)$
  - A graph G is *k-colorable* if it has a proper *k-coloring*
  - *"Proper coloring" and just "coloring" are used interchangeably sometimes*
- The *chromatic number* $\chi(G)$ is the minimum *k* such that G is *k*-colorable
  - If $\chi(G) = k$, then G is *k-chromatic*
  - If $\chi(G) = k$, but $\chi(H) < k$ for every proper subgraph *H* of *G,* then *G* is *color-critical* or *k-critical*

Lower Bounds for χ

- In a proper coloring, each color class is an independent set
- $\omega(G)$ = clique number of G = the order of the largest complete subgraph of G
  - $\chi(G) \geq \omega(G)$
- If $\alpha(G)$ is the independence number of G, then $\chi(G) \geq n(G)/\alpha(G)$

Algorithm Greedy-Coloring

- The greedy coloring with respect to a vertex ordering $v_1, \ldots, v_n$ of $V(G)$ is obtained by coloring vertices in the order $v_1, \ldots, v_n$, assigning to v_i the smallest indexed color not already used on its lower-indexed neighbors.

Upper Bounds for χ

- $\chi(G) \le \Delta(G) + 1$
- If a graph G has degree sequence $d_1 \ge \ldots \ge d_n$, then $\chi(G) \le 1 + \max_i \min\{ d_i, i-1\}$
- If H is a *k*-critical graph, then $\delta(H) \ge k-1$
- If G is a graph, then $\chi(G) \le 1 + \max_{H \subseteq G} \delta(H)$
- Brooks Theorem:

  If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \le \Delta(G)$.

More results

- If G is an interval graph, then $\chi(G) = \omega(G)$
  - So the lower bound of $\omega(G)$ is tight
- But it is possible to create graphs with arbitrarily high χ while keeping ω constant
  - Mycielski's construction
    - Creates a graph with $\chi = k + 1$ from a graph with $\chi = k$ while keeping ω fixed at 2
    - Can be used repeatedly to create arbitrarily large gap between χ and ω

Mycielski's Construction

- Mycielski found a construction that builds from any given *k*-chromatic triangle-free graph G a *k+1*-chromatic triangle-free supergraph G′.
  - Given G with vertex set $V = \{v_1,\ldots,v_n\}$, add vertices $U = \{u_1,\ldots,u_n\}$ and one more vertex w.
  - Beginning with $G'[V] = G$, add edges to make u_i adjacent to all of $N_G(v_i)$, and then make $N(w) = U$. Note that U is an independent set in G′.

Min/Max Edges in k-chromatic Graphs

- Min edges in *k*-chromatic graph with vertices = $\binom{k}{2}$
- *k*-chromatic graph of n vertices with max no. of edges
  - Turan graph ($T_{n,r}$) : complete *r*-partite graph with *n* vertices such that the difference between the no. of nodes in any two partite sets is at most 1
  - $T_{n,k}$ has the maximum no. of edges

Critical Graphs

- Suppose that G is a graph with $\chi(G) > k$ and that X,Y is a partition of V(G). If G[X] and G[Y] are k-colorable, then the edge cut [X,Y] has at least k edges.
- [Dirac] Every k-critical graph is $k-1$ edge-connected.
  - So every k-chromatic graph is also k-1 edge connected

Chromatic Recurrence

- The function $\chi(G;k)$ counts the mappings $f: V(G) \rightarrow [k]$ that properly color G from the set $[k] = \{1,\ldots,k\}$. In this definition, the *k*-colors need not all be used, and permuting the colors used produces a different coloring.

- If G is a simple graph and $e \in E(G)$, then

$$\chi(G; k) = \chi(G - e; k) - \chi(G.e; k)$$

Edge Coloring

- A *k-edge-coloring* of G is a labeling *f*: E(G) → [k]
  - The labels are *colors*
  - The set of edges with one color is a *color class.*
  - A *k*-edge-coloring is *proper* if edges sharing a vertex have different colors; equivalently, each color class is a matching
  - A graph is *k-edge-colorable* if it has a proper *k*-edge-coloring
  - The *edge-chromatic-number* $\chi'(G)$ of a loop-less graph G is the least *k* such that G is k-edge-colorable

Results

- $\chi'(G) \geq \Delta(G)$.
- If G is a loop-less graph, then $\chi'(G) \leq 2\Delta(G) - 1$.
- If G is bipartite, then $\chi'(G) = \Delta(G)$.

- A regular graph G has a $\Delta(G)$-edge coloring if and only if it decomposes into 1-factors. We say that G is *1-factorable.*

- [Vizing's Theorem, 1964] Every simple graph with maximum degree Δ has a proper Δ+1-edge-coloring.