Probability Primer
CS60077: Reinforcement Learning

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Sep 07, 2020

Agenda

To brush up basics of probability and random variables.

Resources

§ "Probability, Statistics, and Random Processes for Electrical
Engineering", 3rd Edition, Alberto Leon-Garcia - [PSRPEE] - Alberto
Leon-Garcia

§ "Machine Learning: A Probabilistic Perspective", Kevin P. Murphy -
[MLAPP] - Kevin Murphy:

Introduction

§ Probability theory is the study of uncertainty.

§ The mathematical treatise of probability is very sophisticated, and delves into a branch of analysis known as **measure theory**.

§ We, however, will go through only basics of probability theory at a level appropriate for our Reinforcement Learning course.

Introduction

§ Probability is the Mathematical language for quantifying *uncertainty*. The starting point is to specify random experiments, sample space and set of outcomes.

§ A **random experiment** is an experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.

§ An **outcome** is a result of the random experiment and it can not be decomposed in terms of other results. The **sample space** of a random experiment is defined as the set of all possible outcomes. An outcome and the sample space of a random experiment will be denoted as ζ and S respectively.

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§ Examples of random experiment

▶ Flipping a coin

▶ Rolling a die

▶ Flipping a coin twice

▶ Pick a number X at random between zero and one, then pick a number Y at random between zero and X.

§ The corresponding sample spaces will be

▶ $S_1 = \{H, T\}$

▶ $S_2 = \{1, 2, 3, 4, 5, 6\}$

▶ $S_3 = \{HH, HT, TH, TT\}$

▶ $S_4 = \{(x, y) : 0 \le y \le x \le 1\}$.

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§ Any subset E of the sample space S is known as an **event**. We, sometimes, are not interested in the occurrence of specific outcomes but rather in the occurrence of a combination of a few outcomes. This requires that we consider subsets of S

▶ Getting even number when rolling a die, $E_2 = \{2, 4, 6\}$

▶ Number of heads equal to number of tails when flipping a coin twice, $E_3 = \{HT, TH\}$

▶ Two numbers differ by less than $1/10$, $E_4 = \{(x, y) : 0 \le y \le x \le 1 \text{ and } |x - y| < 1/10\}$.

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§ Three events are of special importance.

- ▶ **Simple event** are the outcomes of random experiments.
- ▶ **Sure event** is the sample space S which consists of all outcomes and hence always occurs.
- ▶ **Impossible** or **null event** ϕ which contains no outcomes and hence never occurs.

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§ **Set of events** (or **event space**) $\mathcal{F}$: A set whose elements are subsets of the sample space (*i.e.*, events). $\mathcal{F} = \{A : A \subseteq S\}$. $\mathcal{F}$ is really a "set of sets".

§ $\mathcal{F}$ should satisfy the following three properties.

- ▶ $\phi \in \mathcal{F}$
- ▶ $A \in \mathcal{F} \implies A^c (\triangleq S \setminus A) \in \mathcal{F}$
- ▶ $A_1, A_2, \cdots \in \mathcal{F} \implies \cup_i A_i \in \mathcal{F}$

Introduction

§ Probabilities are numbers assigned to events of $\mathcal{F}$ that indicate how "likely" it is that the events will occur when a random experiment is performed.

§ Let a random experiment has sample space S and event space $\mathcal{F}$. Probability of an event A is a function $P : \mathcal{F} \to \mathbb{R}$ that satisfies the following properties

▶ $P(A) \geq 0, \ \forall A \in \mathcal{F}$

▶ $P(S) = 1$

▶ If $A_1, A_2, \cdots \in \mathcal{F}$ are disjoint events (*i.e.*, $A_i \cap A_j = \phi$ for $i \neq j$) then, $P(\cup_i A_i) = \sum_i P(A_i)$

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§ Properties

- ▶ $P(A^c) = 1 - P(A)$
- ▶ $P(A) \leq 1$
- ▶ $P(\phi) = 0$
- ▶ If $A \subseteq B$, then $P(A) \leq P(B)$.
- ▶ $P(A \cap B) \leq \min(P(A), P(B))$
- ▶ $P(A \cup B) \leq P(A) + P(B)$

Conditional Probability

§ **Conditional probability** provides whether two events are related in the sense that knowledge about the occurrence of one, say B, alters the likelihood of occurrence of the other say, A.

§ This conditional probability is defined as,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

§ Two events A and B are **independent** (denoted as $A \perp B$) if the knowledge of occurrence of one does not change the likelihood of occurrence of the other. This translates to the condition for independence as,

$$P(A|B) = P(A)$$
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Total Probability Theorem

§ Let $B_1, B_2, \cdots, B_n$ be exhaustive and mutually exclusive events such that each of these events has positive probabilities. Then for any event A, the *total probability theorem* says,

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i) \tag{1}$$

§ **Proof:** Since, $B_1, B_2, \cdots, B_n$ are exhaustive (*i.e.*, their union covers the whole sample space), $A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots (A \cap B_n)$

$$P(A) = P((A \cap B_1) \cup (A \cap B_2) \cup \cdots (A \cap B_n))$$
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\end{aligned}
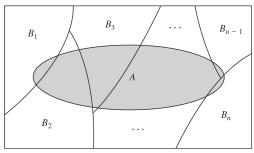$$

Total Probability Theorem

Figure credit: [PSRPEE] - Alberto Leon-Garcia

§ This is also known as **marginalization** operation.

§ Such exhaustive and mutually exclusive events $B_1, B_2, \cdots, B_n$ are also said to form a **partition** of the sample space.

Bayes Rule

§ The total probability theorem is often used in conjunction with the Bayes' Rule that relates conditional probabilities of the form $P(B|A)$ with conditional probabilities of the form $P(A|B)$.

§ Let the events $B_1, B_2, \cdots, B_n$ partitions a sample space such that each of the $P(B_i)$'s are non-negative. The Bayes' rule states,

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum\limits_{i=1}^{n} P(A|B_i)P(B_i)} \tag{2}$$

§ Bayes' rule is a very important tool for inference in machine learning. A can be thought of as the "effect" and B_i's are several "causes" that can result in the effect. From the probabilities of the causes (B_i's) resulting in the effect (A) and the probability of the causes (B_i's) to occur frequently, the probability that a particular cause (B_i) is the reason behind the effect (A) is computed.

Random Variables

§ Statistics and Machine Learning are concerned with data. The link to sample space and events to data is **Random Variables**.

§ A random variable is a mapping ($X : S \to \mathbb{R}$) from the sample space to real values that assigns a real number ($X(\zeta)$) to each outcome (ζ) in the sample space of a random experiment.
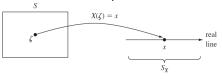

FIGURE 3.1
A random variable assigns a number $X(\zeta)$ to each outcome ζ in the sample space S of a random experiment.
Figure credit: [PSRPEE] - Alberto Leon-Garcia

§ We will use the following notation: capital letters denote random variables, *e.g.*, X or Y, and lower case letters denote possible values of the random variables, *e.g.*, x or y.

Random Variables

§ An example from [PSRPEE] - Alberto Leon-Garcia

Example 3.1   Coin Tosses

A coin is tossed three times and the sequence of heads and tails is noted. The sample space for this experiment is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Let X be the number of heads in the three tosses. X assigns each outcome ζ in S a number from the set $S_X = \{0, 1, 2, 3\}$. The table below lists the eight outcomes of S and the corresponding values of X.

ζ:	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
$X(\zeta)$:	3	2	2	2	1	1	1	0

X is then a random variable taking on values in the set $S_X = \{0, 1, 2, 3\}$.

§ Since the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

$$P(X = x) = P(\{\zeta \in S; \ X(\zeta) = x\}) \qquad (3)$$

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$P[X = 0] = P[\{TTT\}] = \dfrac{1}{8}$

$P[X = 1] = P[\{HTT, THT, TTH\}] = P[\{HTT\}] + P[\{THT\}] + P[\{TTH\}] = \dfrac{3}{8}$

$P[X = 2] = P[\{HHT, HTH, THH\}] = P[\{HHT\}] + P[\{HTH\}] + P[\{THH\}] = \dfrac{3}{8}$

$P[X = 3] = P[\{HHH\}] = \dfrac{1}{8}$

Plot generated by: discrete_prob_dist_plot

from [MLAPP] - Kevin Murphy

Discrete Random Variables and PMF

§ A **discrete random variable** X is defined as a random variable that can take at most a countable number of possible values, *i.e.*, $S_X = \{x_1, x_2, x_3, \cdots\}$.

§ A discrete random variable is said to be **finite** if its range is finite, *i.e.*, $S_X = \{x_1, x_2, x_3, \cdots, x_n\}$.

§ The probabilities of events involving a discrete random variable X forms the **Probability Mass Function (PMF)** of X and it is defined as (ref eqn. (3)),

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Continuous Random Variables and PDF

§ Random variables with a continuous range of possible experimental values are quite common.

§ X is a **continuous random variable** if there exists a non-negative function $f_X(x)$, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers, $P(X \in B) = \int_B f_X(x)dx$. The function $f_X(x)$ is called the **probability density function (PDF)** of the random variable X.

§ Some properties of PDFs

▶ $P(-\infty < X < \infty) = \int_{\infty}^{\infty} f_X(x)dx = 1$

▶ $P(a \leq X \leq b) = \int_a^b f_X(x)dx$

▶ If we let $a = b$ in the preceding, then $P(X = a) = \int_a^a f_X(x)dx = 0$

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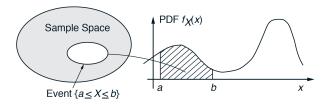
Continuous Random Variables and PDF

Figure 3.1: Illustration of a PDF. The probability that X takes value in an interval $[a, b]$ is $\int_a^b f_X(x)\, dx$, which is the shaded area in the figure.
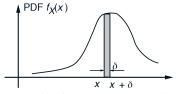Fig credit: MIT Course: 6.041-6.43, Lecture Notes

Figure 3.2: Interpretation of the PDF $f_X(x)$ as "probability mass per unit length" around x. If δ is very small, the probability that X takes value in the interval $[x, x + \delta]$ is the shaded area in the figure, which is approximately equal to $f_X(x) \cdot \delta$.

Fig credit: MIT Course: 6.041-6.43, Lecture Notes

Cumulative Distribution Function

§ We have defined PMF and PDF for discrete and continuous random variables respectively.

§ **Cumulative Distribution Function (CDF)** is a concept that is applicable to both discrete and continuous random variables. It is defined as,

$$F_X(x) = P(X \leq x) = \begin{cases} \sum\limits_{k \leq x} P_X(k) & X : \text{ discrete} \\ \int\limits_{-\infty}^{x} f_X(t)dt & X : \text{ continuous} \end{cases} \tag{5}$$

§ For continuous random variables, the cumulative distribution function $F_X(x)$ is differentiable everywhere. Naturally, in these cases, PDF is the derivative of the CDF.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

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$$f_X(x) = \frac{dF_X(x)}{dx}$$

Cumulative Distribution Function

Cumulative Distribution Function

Fig credit: MIT Course: 6.041-6.43, Lecture Notes

CDF - Some Properties

§ $0 \leq F_X(x) \leq 1$

§ $\lim_{x \to -\infty} F_X(x) = 0$

§ $\lim_{x \to \infty} F_X(x) = 1$

§ $x \leq y \implies F_X(x) \leq F_X(y)$

Expectation

§ The **expected value/expectation/mean** of a random variable is
defined as:

$$\mathbb{E}[X] = \begin{cases} \sum_x x P_X(x) & \text{when } X \text{ is discrete} \\ \int x f_X(x) dx & \text{when } X \text{ is continuous} \end{cases} \tag{6}$$

§ **Functions of random variable**: If $Y = g(X)$ is a function of a
random variable X, then Y is also a random variable, since it provides
a numerical value for each possible outcome.

§ For a function of the random variable $Y = g(X)$, the expectation is,
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Variance

§ $\mathbb{E}[X]$ is also referred to as the **first moment** of X. Similarly the second moment is defined as $\mathbb{E}[X^2]$ and in general, the nth moment as $\mathbb{E}[X^n]$

§ Another quantity of interest is the variance of a random variable x, denoted as $\text{var}(X)$ and defined as $\mathbb{E}[(X - \mathbb{E}[X])^2]$. Variance provides a measure of dispersion of X around its mean $\mathbb{E}[X]$.

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Properties

§ Expectation

- ▶ $\mathbb{E}[a] = a$ for any constant $a \in \mathbb{R}$
- ▶ $\mathbb{E}[af(X)] = a\mathbb{E}[f(X)]$ for any constant $a \in \mathbb{R}$
- ▶ $\mathbb{E}[f(X) + g(X)] = \mathbb{E}[f(X)] + \mathbb{E}[g(X)]$

§ Variance

- ▶ $\mathrm{var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right] = \mathbb{E}[X^2] - \left[\mathbb{E}[X]\right]^2$
- ▶ $\mathrm{var}(a) = 0$ for any constant $a \in \mathbb{R}$
- ▶ $\mathrm{var}(af(X)) = a^2 \, \mathrm{var}(f(X))$ for any constant $a \in \mathbb{R}$

Some Common Random Variables

Discrete Random Variables

§ **Bernoulli** random variable: Takes two values 1 and 0 (or 'Head' and 'Tail'). The PMF is given by,

$$P_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases} \tag{9}$$

This is also written as $P_X(x) = p^x(1-p)^{1-x}$

§ It is used to model situations with just two random outcomes *e.g.*, tossing a coin once.

§ For $X \sim \text{Ber}(p), \mathbb{E}(X) = p$ and $\text{var}(X) = p(1-p)$.

Some Common Random Variables

Discrete Random Variables

§ **Binomial** random variable: is used to model more complex situation *e.g.*, the number of heads if a coin is tossed n times. The PMF is given by,

$$P_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \cdots, n. \quad (10)$$

§ For $X \sim \text{Bin}(n, p), \mathbb{E}(X) = np$ and $\text{var}(X) = np(1-p)$.

Some Common Random Variables

Discrete Random Variables

§ **Poisson** random variable: models situations where the events occur completely at random in time or space. The random variable counts the number of occurrences of the event in a certain time period or in a certain region in space. The PMF is given by,

$$P_X(x) = P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, 2, \cdots \tag{11}$$

where λ is the average number of occurrences of the event in that specified time interval or region in space.

§ For $X \sim \text{Poisson}(\lambda), \mathbb{E}(X) = \lambda$ and $\text{var}(X) = \lambda$.

Some Common Random Variables

Continuous Random Variables

§ **Uniform** random variable: X is a uniform random variable on the interval (a, b) if its probability density function is given by,

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

§ For $X \sim \text{Uniform}(a, b)$, $\mathbb{E}(X) = \frac{a+b}{2}$ and $\text{var}(X) = \frac{(b-a)^2}{12}$.

Fig credit: MIT Course: 6.041-6.43, Lecture Notes

Some Common Random Variables

Continuous Random Variables

§ **Exponential** random variable: X is a exponential random variable if its probability density function is given by,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

§ For $X \sim \text{Exponential}(\lambda), \mathbb{E}(X) = \frac{1}{\lambda}$ and $\text{var}(X) = \frac{1}{\lambda^2}$.

Fig credit: MIT Course: 6.041-6.43, Lecture Notes

Some Common Random Variables

Continuous Random Variables

§ **Gaussian/Normal** random variable: X is a Gaussian/Normal random variable if its probability density function is given by,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{14}$$

§ For $X \sim \text{Gaussian}(\mu, \sigma^2), \mathbb{E}(X) = \mu$ and $\text{var}(X) = \sigma^2$.

§ Gaussianity is Preserved by Linear Transformations. If $X \sim \text{Gaussian}(\mu, \sigma^2)$ and if a, b are scalars, the random variable $Y = aX + b$ is also Gaussian with mean and variance $\mathbb{E}(X) = a\mu + b$ and $\text{var}(X) = a^2\sigma^2$ respectively.

Two Random Variables

§ Many random experiments involve several random variables. For example, temperature and pressure of a room during different days.

Figure credit: [PSRPEE] - Alberto Leon-Garcia

§ Consider two discrete random variables X and Y associated with the same experiment. We will use the notation $P(X = x, Y = y)$ to denote $P(X = x$ and $Y = y)$.

Two Random Variables

§ The **Joint PMF** of the two random variables X and Y is defined as,

$$
\begin{aligned}
P_{X,Y}(x,y) &= P(X = x, Y = y) \\
&= P(\{\zeta \in S; \, X(\zeta) = x, Y(\zeta) = y\} \text{ for real } x \text{ and } y)
\end{aligned}
\tag{15}
$$

§ $P_X(x)$ and $P_Y(y)$ are sometimes referred to as the **marginal PMFs**, to distinguish them from the joint PMF.

§ The marginal and the joint PMFs are related in the following way (ref eqn. (1), the total probability theorem),

$$
P_X(x) = \sum_y P_{X,Y}(x,y) \text{ and } P_Y(y) = \sum_x P_{X,Y}(x,y)
\tag{16}
$$

Two Random Variables

§ Similar to PDFs for single random variable, **joint PDF** for two continuous random variables is defined. for sets A and B of real numbers,

$$P(X \in A, Y \in B) = \int_B \int_A f_{X,Y}(x,y)dxdy \tag{17}$$

§ Similarly, **joint CDF** is also defined.

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \begin{cases} \sum\limits_{l \le y} \sum\limits_{k \le x} P_{X,Y}(k,l) & X, Y : \text{discrete} \\ \int\limits_{-\infty}^{y} \int\limits_{-\infty}^{x} f_{X,Y}(u,v)dudv & X, Y : \text{continuous} \end{cases}$$

$$\tag{18}$$

§ Differentiation for continuous random variables, yields

$$f_{X,Y}(x,y) = \frac{dF_{X,Y}(x,y)}{dydx}$$

Some Useful Relations

§ Marginal CDF can be obtained by setting the value of the other Random Variable to ∞, *i.e.*, $F_X(x) = F_{X,Y}(x, \infty)$ and $F_Y(y) = F_{X,Y}(\infty, y)$.

§ Similar relations exist between marginal and joint PDFs.
$f_X(x) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dy$ and $f_Y(y) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

§ Conditional PMF and Marginal PMF for discrete variables are related as, $P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$ assuming that $P_X(x) \neq 0$.

§ Similar relation is there for continuous random variables.
$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$ provided $f_X(x) \neq 0$.

Joint Expectations

§ Similar expectation and moment rules exist for joint moments and expectation as in the case of a single random variable.

§ Considering $Z = g(X, Y)$ as a function of two random variables, the expectation of Z can be found as,

$$
\mathbb{E}[Z] = \begin{cases} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy & X, Y \text{ continuous} \\ \sum\limits_{i} \sum\limits_{j} g(x_i, y_j) P_{X,Y}(x_i, y_n) & X, Y \text{ discrete} \end{cases} \tag{19}
$$

§ Expectation of a sum of random variables is the sum of the expectations of the random variables.

$$
\mathbb{E}[X_1 + X_2 + X_3 + \cdots] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \cdots \tag{20}
$$

Joint Moments, Correlation, and Covariance

§ The jk^{th} **joint moment** of X and Y is defined as,

$$\mathbb{E}[X^j Y^k] = \begin{cases} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x^j y^k f_{X,Y}(x,y)dxdy & X, Y \text{ continuous} \\ \sum\limits_{m} \sum\limits_{n} x_m^j y_n^k P_{X,Y}(x_m, y_n) & X, Y \text{ discrete} \end{cases} \tag{21}$$

§ When $j = k = 1$, the corresponding moment $\mathbb{E}[XY]$ gives the correlation between X and Y. If $\mathbb{E}[XY] = 0$, X and Y are said to be **orthogonal**.

§ The jk^{th} **central moment** of X and Y is defined as
$\mathbb{E}\left[(X - \mathbb{E}(X))^j (Y - \mathbb{E}(Y))^k\right]$

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Joint Moments, Correlation, and Covariance

§ Covariance can also be expressed as $\text{COV}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

§ If X and Y are independent, then $\text{COV}(X, Y) = 0$, *i.e.*, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

§ Correlation coefficient turns covariance into a normalized scale between -1 to 1.

$$\rho_{X,Y} = \frac{\text{COV}(X, Y)}{\sqrt{\text{VAR}(X)}\sqrt{\text{VAR}(Y)}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{VAR}(X)}\sqrt{\text{VAR}(Y)}} \tag{22}$$

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Joint Moments, Correlation, and Covariance

§ For example, let $X \sim \mathcal{U}(-1, 1)$ and $Y = X^2$. Clearly, Y is dependent on X, but it can be shown that $\rho_{X,Y} = 0$.

$$\mathbb{E}[X] = \frac{-1 + 1}{2} = 0, \text{VAR}[X] = \frac{\left(1 - (-1)\right)^2}{12} = \frac{1}{3}$$

$$\mathbb{E}[Y] = \mathbb{E}[X^2] = \text{VAR}[X] + \left(\mathbb{E}[X]\right)^2 = \frac{1}{3} - 0^2 = \frac{1}{3}$$

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§ If X and Y are independent random variables, then random variables defined by any pair of functions $g(X)$ and $h(Y)$ are also independent, i.e., if $P(XY) = P(X)P(Y)$ then $P(g(X)h(Y)) = P(g(X))P(h(y))$.

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Conditional Expectation

§ The conditional expectation of Y given $X = x$ is defined as,

$$\mathbb{E}[Y|x] = \int\limits_{-\infty}^{\infty} y f_{Y|x}(y|x) dy \tag{24}$$

§ The conditional expectation $\mathbb{E}(Y|x)$ can be viewed as defining a function of x, $g(x) = \mathbb{E}(Y|x)$. As x, is a result of a random experiment, $\mathbb{E}(Y|x)$ is a random variable. So, we can find its expectation as,

$$\mathbb{E}\big[\mathbb{E}[Y|x]\big] = \int\limits_{-\infty}^{\infty} \mathbb{E}[Y|x] f_X(x) dx = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} y f_{Y|x}(y|x) f_X(x) dx dy \tag{25}$$

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§ X and Y are **conditionally independent** given Z iff the conditional joint can be written as product of conditional marginals,

$$X \perp\!\!\!\perp Y | Z \Leftrightarrow P(X, Y | Z) = P(X|Z)P(Y|Z) \qquad (26)$$

§ Conditional independence also implies,

$$X \perp\!\!\!\perp Y | Z \Rightarrow P(X|Y, Z) = P(X|Z) \text{ and } P(Y|X, Z) = P(Y|Z) \qquad (27)$$

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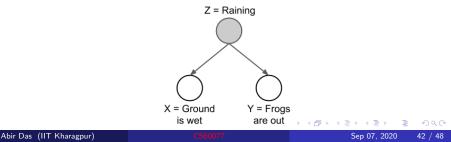$$X \perp\!\!\!\perp Y | Z \Leftrightarrow P(X, Y | Z) = P(X|Z)P(Y|Z) \qquad (26)$$

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Z = Raining

X = Ground
is wet

Y = Frogs
are out

Multiple Random Variables

§ The notions and ideas can be generalized to more than two random variables. A **vector random variable X** is a function that assigns a vector of real numbers to each outcome ζ in the sample space S of a random experiment.

§ Uppercase boldface letters are generally used to denote vector random variables. By convention, it is a column vector. Each X_i can be thought of as a random variable itself.

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} X_1, X_2, \cdots, X_n \end{bmatrix}^T$$

§ Possible values of the vector random variable are denoted by
$\mathbf{x} = \begin{bmatrix} x_1, x_2, \cdots, x_n \end{bmatrix}^T$

Multiple Random Variables

§ The **Joint PMF** of n-dimensional discrete random vector $\mathbf{X}$

$$P_{\mathbf{X}}(\mathbf{x}) = P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) \tag{28}$$

§ Relation between the marginal and the joint PMFs,

$$P_{X_1}(x_1) = \sum_{x_2} \cdots \sum_{x_n} P_{\mathbf{X}}(\mathbf{x}) \tag{29}$$

§ Similarly, **joint CDF** is also defined.

$$F_{\mathbf{X}}(\mathbf{x}) = P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n)$$

$$= \begin{cases} \sum\limits_{j \leq x_1} \sum\limits_{k \leq x_2} \cdots \sum\limits_{l \leq x_n} P_{\mathbf{X}}([x_1, x_2, \cdots, x_n]^T) & \mathbf{X} : \text{ discrete} \\ \int\limits_{-\infty}^{x_1} \int\limits_{-\infty}^{x_2} \cdots \int\limits_{-\infty}^{x_n} f_{\mathbf{x}}([u, v, \cdots, w]^T) du\, dv \cdots dw & \mathbf{X} : \text{ continuous} \end{cases}$$

$$\tag{30}$$

Multiple Random Variables

§ The **joint PDF** of n-dimensional continuous random vector $\mathbf{X}$

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\partial^n F_{\mathbf{X}}(\mathbf{x})}{\partial x_1 \partial x_2 \cdots \partial x_3} \tag{31}$$

§ The **marginal PDF**

$$f_{X_1}(x_1) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} f_{\mathbf{x}}([x_1, x_2, x_3, \cdots, x_n]^T) \, dx_2 dx_3 \cdots dx_n \tag{32}$$

§ The **conditional PDF**

$$f_{X_1/X_2,\cdots,X_n}(x_1/x_2,\cdots,x_n) = \frac{f_{\mathbf{X}}(\mathbf{x})}{f_{X_2,\cdots,X_n}(x_2,\cdots,x_n)} \tag{33}$$

§ **Chain rule**

$$f(x_1, x_2, \cdots, x_n) = f(x_n|x_1,\cdots,x_{n-1})f(x_1,\cdots,x_{n-1})$$
$$= f(x_n|x_1,\cdots,x_{n-1})f(x_{n-1}|x_1,\cdots,x_{n-2})f(x_1,\cdots,x_{n-2})$$
$$= f(x_1)\prod_{i=2}^{n} f(x_i|x_1, x_2, \cdots, x_{i-1})$$

Multiple Random Variables

§ There's also natural generalization of **independence**.

$$f(x_1, x_2, \cdots, x_n) = f(x_1)f(x_2)\cdots f(x_n) \tag{35}$$

§ **Expectation**: Consider an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}$. The expected value is,

$$\mathbb{E}\big[g(\mathbf{X})\big] = \int\limits_{\mathbb{R}^n} g(\mathbf{X}) f_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} \tag{36}$$

§ If $\mathbf{g}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, then the expected value of $\mathbf{g}$ is the element-wise expected values of the output vector, *i.e.*, if $\mathbf{g}(\mathbf{x}) = \big[g_1(\mathbf{x}), g_2(\mathbf{x}, \cdots, g_n(\mathbf{x}))\big]^T$, then

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Multiple Random Variables

§ **Covariance matrix**: For a random vector $\mathbf{X} \in \mathbb{R}^n$, covariance matrix $\mathbf{\Sigma}$ is $n \times n$ square matrix whose entries are given by $\mathbf{\Sigma}_{ij} = \mathsf{Cov}(X_i, X_j)$.

$$\mathbf{\Sigma} = \begin{bmatrix} \mathsf{Var}(X_1, X_1) & \mathsf{Cov}(X_1, X_2) & \cdots & \mathsf{Cov}(X_1, X_n) \\ \mathsf{Cov}(X_2, X_1) & \mathsf{Var}(X_2, X_2) & \cdots & \mathsf{Var}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{Cov}(X_n, X_1) & \mathsf{Cov}(X_n, X_2) & \cdots & \mathsf{Var}(X_n, X_n) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{E}[X_1^2] - \mathbb{E}[X_1]\mathbb{E}[X_1] & \cdots & \mathbb{E}[X_1 X_n] - \mathbb{E}[X_1]\mathbb{E}[X_n] \\ \mathbb{E}[X_2 X_1] - \mathbb{E}[X_2]\mathbb{E}[X_1] & \cdots & \mathbb{E}[X_2 X_n] - \mathbb{E}[X_2]\mathbb{E}[X_n] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[X_n X_1] - \mathbb{E}[X_n]\mathbb{E}[X_1] & \cdots & \mathbb{E}[X_n^2] - \mathbb{E}[X_n]\mathbb{E}[X_n] \end{bmatrix}$$

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$$= \mathbb{E}[\mathbf{X}\mathbf{X}^T] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{X}^T] = \mathbb{E}\big[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T\big]$$

Linear Transformations of Random Vectors

§ Suppose $\mathbf{X}$ is some random vector and $\mathbf{Y} = \mathbf{f}(\mathbf{X})$, then we would like to know what are the first two moments of Y.

§ Let $\mathbf{f}(.)$ is a linear function that is $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where $\mathbf{X} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^m$.

§ The mean will be $\mathbb{E}[\mathbf{Y}] = \mathbb{E}[\mathbf{A}\mathbf{X} + \mathbf{b}] = \mathbf{A}\mathbb{E}[\mathbf{X}] + \mathbf{b}$.

§ The covariance matrix $\mathbf{\Sigma_Y}$ is given by,

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\begin{aligned}
\mathbf{\Sigma_Y} &= \mathbb{E}\big[(\mathbf{Y} - \mathbb{E}[\mathbf{Y}])(\mathbf{Y} - \mathbb{E}[\mathbf{Y}])^T\big] \\
&= \mathbb{E}\big[(\mathbf{A}\mathbf{X} + \mathbf{b} - \mathbf{A}\mathbb{E}[\mathbf{X}] - \mathbf{b})(\mathbf{A}\mathbf{X} + \mathbf{b} - \mathbf{A}\mathbb{E}[\mathbf{X}] - \mathbf{b})^T\big] \\
&= \mathbb{E}\big[\mathbf{A}(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T\mathbf{A}^T\big] \\
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\end{aligned}
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§ **Cross-covariance** between $\mathbf{X}$ and $\mathbf{Y}$ is $\mathbf{\Sigma_{XY}} = \mathbb{E}\big[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{Y} - \mathbb{E}[\mathbf{Y}])^T\big]$

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