# Algebraic Curves

## An Elementary Introduction


Abhijit Das

Department of Computer Science and Engineering
Indian Institute of Technology Kharagpur


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# Part I

## Affine and Projective Curves

- Rational Points on Curves
- Polynomial and Rational Functions on Curves
- Divisors and Jacobians on Curves

# Affine Curves

- $K$ is a field.
- $\overline{K}$ is the algebraic closure of $K$.
- It is often necessary to assume that $K$ is algebraically closed.
- **Affine plane:** $K^2 = \{(h, k) \mid h, k \in K\}$.
- For $(h, k) \in K^2$, the field elements $h, k$ are called **affine coordinates**.
- **Affine curve:** Defined by a polynomial equation:

$$C : f(X, Y) = 0.$$

- It is customary to consider only irreducible polynomials $f(X, Y)$. If $f(X, Y)$ admits non-trivial factors, the curve $C$ is the set-theoretic union of two (or more) curves of smaller degrees.
- **Rational points on $C$:** All points $(h, k) \in K^2$ such that $f(h, k) = 0$.
- Rational points on $C$ are called **finite points**.

# Affine Curves: Examples

- **Straight lines:** $aX + bY + c = 0$.
- **Circles:** $(X - a)^2 + (Y - b)^2 - r^2 = 0$.
- **Conic sections:** $aX^2 + bXY + cY^2 + dX + eY + f = 0$.
- **Elliptic curves:** Defined by the *Weierstrass equation*:
  $Y^2 + (a_1 X + a_3)Y = X^3 + a_2 X^2 + a_4 X + a_6$.
  If $\operatorname{char} K \neq 2, 3$, this can be simplified as $Y^2 = X^3 + aX + b$.
- **Hyperelliptic curves of genus $g$:** $Y^2 + u(X)Y = v(X)$ with $\deg u \leqslant g$, $\deg v = 2g + 1$, and $v$ monic.
  If $\operatorname{char} K \neq 2$, this can be simplified as $Y^2 = w(X)$ with $\deg w = 2g + 1$ and $w$ monic.
- Parabolas are hyperelliptic curves of genus 0.
- Elliptic curves are hyperelliptic curves of genus 1.

# Projective Plane

- Define a relation $\sim$ on $K^3 \setminus \{(0,0,0)\}$ as $(h,k,l) \sim (h',k',l')$ if $h' = \lambda h$, $k' = \lambda k$ and $l' = \lambda l$ for some non-zero $\lambda \in K$.
- $\sim$ is an equivalence relation on $K^3 \setminus \{(0,0,0)\}$.
- The equivalence class of $(h,k,l)$ is denoted by $[h,k,l]$.
- $[h,k,l]$ can be identified with the line in $K^3$ passing through the origin and the point $(h,k,l)$.
- The set of all these equivalence classes is the **projective plane** over $K$.
- The projective plane is denoted as $\mathbb{P}^2(K)$.
- $h,k,l$ in $[h,k,l]$ are called **projective coordinates**.
- Projective coordinates are unique up to multiplication by non-zero elements of $K$.
- The three projective coordinates cannot be simultaneously 0.

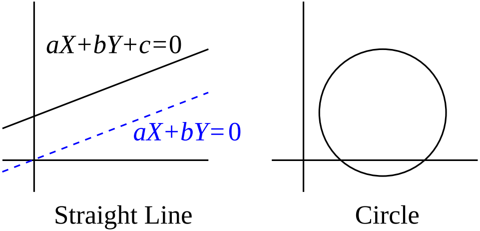
# Relation Between the Affine and the Projective Planes

- $\mathbb{P}^2(K)$ is the affine plane $K^2$ plus the points at infinity.
- Take $P = [h, k, l] \in \mathbb{P}^2(K)$.
- **Case 1:** $l \neq 0$.
  - $P = [h/l, k/l, 1]$ is identified with the point $(h/l, k/l) \in K^2$.
  - The line in $K^3$ corresponding to $P$ meets $Z = 1$ at $(h/l, k/l, 1)$.
  - $P$ is called a **finite point**.
- **Case 2:** $l = 0$.
  - The line in $K^3$ corresponding to $P$ does not meet $Z = 1$.
  - $P$ does not correspond to a point in $K^2$.
  - $P$ is a **point at infinity**.
  - For every slope of lines in the $X, Y$-plane, there exists exactly one point at infinity.
- A line passes through all the points at infinity. It is the **line at infinity**.
- Two distinct lines (parallel or not) in $\mathbb{P}^2(K)$ always meet at a unique point (consistent with Bézout's theorem).
- Through any two distinct points in $\mathbb{P}^2(K)$ passes a unique line.

# Passage from Affine to Projective Curves

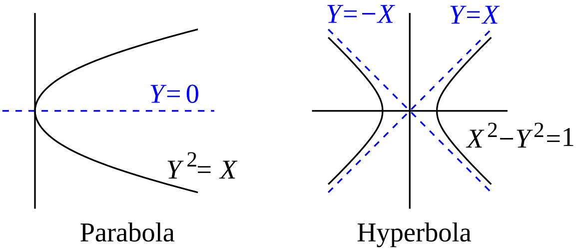
- A (multivariate) polynomial is called **homogeneous** if every non-zero term in the polynomial has the same degree.
- Example: $X^3 + 2XYZ - 3Z^3$ is homogeneous of degree 3. $X^3 + 2XY - 3Z$ is not homogeneous. The zero polynomial is homogeneous of any degree.
- Let $C : f(X, Y) = 0$ be an affine curve of degree $d$.
- $f^{(h)}(X, Y, Z) = Z^d f(X/Z, Y/Z)$ is the **homogenization** of $f$.
- $C^{(h)} : f^{(h)}(X, Y, Z) = 0$ is the **projective curve** corresponding to $C$.
- For any non-zero $\lambda \in K$, we have $f^{(h)}(\lambda h, \lambda k, \lambda l) = \lambda^d f^{(h)}(h, k, l)$. So $f^{(h)}(\lambda h, \lambda k, \lambda l) = 0$ if and only if $f^{(h)}(h, k, l) = 0$.
- The rational points of $C^{(h)}$ are all $[h, k, l]$ with $f^{(h)}(h, k, l) = 0$.
- **Finite points on $C^{(h)}$:** Put $Z = 1$ to get $f^{(h)}(X, Y, 1) = f(X, Y)$. These are the points on $C$.
- **Points at infinity on $C^{(h)}$:** Put $Z = 0$ and solve for $f^{(h)}(X, Y, 0) = 0$. These points do not belong to $C$.

# Examples of Projective Curves



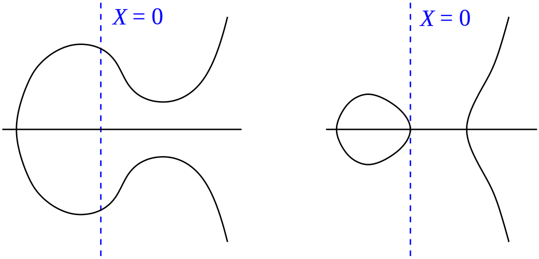

- **Straight line:** $aX + bY + cZ = 0$.
  - Finite points: Solutions of $aX + bY + c = 0$.
  - Points at infinity: Solve for $aX + bY = 0$.
    If $b \neq 0$, we have $Y = -(a/b)X$. So $[1, -(a/b), 0]$ is the only point at infinity.
    If $b = 0$, we have $aX = 0$, that is, $X = 0$. So $[0, 1, 0]$ is the only point at infinity.
- **Circle:** $(X - aZ)^2 + (Y - bZ)^2 = r^2Z^2$.
  - Finite points: Solutions of $(X - a)^2 + (Y - b)^2 = r^2$.
  - Points at infinity: Solve for $X^2 + Y^2 = 0$.
    For $K = \mathbb{R}$, the only solution is $X = Y = 0$, so there is no point at infinity.
    For $K = \mathbb{C}$, the solutions are $Y = \pm \mathrm{i}X$, so there are two points at infinity: $[1, \mathrm{i}, 0]$ and $[1, -\mathrm{i}, 0]$.

# Examples of Projective Curves (contd.)




- **Parabola:** $Y^2 = XZ$.
  - Finite points: Solutions of $Y^2 = X$.
  - Points at infinity: Solve for $Y^2 = 0$.
    $Y = 0$, so $[1, 0, 0]$ is the only point at infinity.
- **Hyperbola:** $X^2 - Y^2 = Z^2$.
  - Finite points: Solutions of $X^2 - Y^2 = 1$.
  - Points at infinity: Solve for $X^2 - Y^2 = 0$.
    $Y = \pm X$, so there are two points at infinity: $[1, 1, 0]$ and $[1, -1, 0]$.

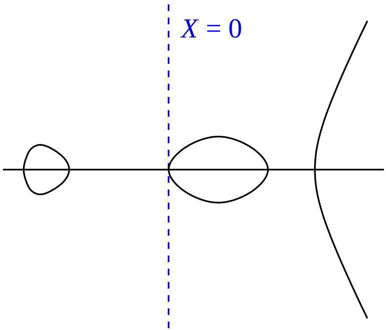
# Examples of Projective Curves (contd.)




$Y^2 = X^3 - X + 1$ $\qquad$ $Y^2 = X^3 - X$

- **Elliptic curve:** $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$.
  - Finite points: Solutions of $Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$.
  - Points at infinity: Solve for $X^3 = 0$.
    $X = 0$, that is, $[0, 1, 0]$ is the only point at infinity.

# Examples of Projective Curves (contd.)

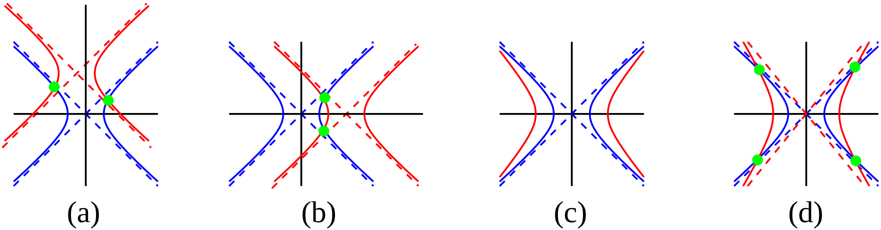



A hyperelliptic curve of genus 2: $Y^2 = X(X^2 - 1)(X^2 - 2)$

- **Hyperelliptic curve:** $Y^2 Z^{2g-1} + Z^g u(X/Z) Y Z^g = Z^{2g+1} v(X/Z)$.
  - Finite points: Solutions of $Y^2 + u(X)Y = v(X)$.
  - Points at infinity: The only $Z$-free term is $X^{2g+1}$ (in $Z^{2g+1} v(X/Z)$). So $[0, 1, 0]$ is the only point at infinity.

# Bézout's Theorem

- A curve of degree $m$ and a curve of degree $n$ intersect at exactly $mn$ points.
- The intersection points must be counted with proper multiplicity.
- It is necessary to work in algebraically closed fields.
- Still, the theorem is not true. For example, two parallel lines or two concentric circles never intersect.
- Passage to the projective plane makes Bézout's theorem true.




(a) and (b): Two simple intersections at the points at infinity
(c): Two tangents at the points at infinity
(d): No intersections at the points at infinity

# Smooth Curves

Let $C : f(X, Y, Z) = 0$ be a projective curve, and $P = [h, k, l]$ a rational point on $C$.

- $P$ is called a **smooth point** on $C$ if the tangent to $C$ at $P$ is uniquely defined.
- **Case 1:** $P$ is a finite point.

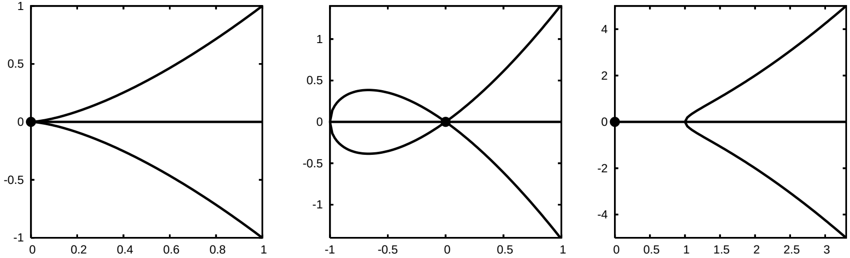
  Now, $l \neq 0$. Consider the affine equation $f(X, Y) = 0$.
  Both $\frac{\partial f}{\partial X}$ and $\frac{\partial f}{\partial Y}$ do not vanish simultaneously at $(h/l, k/l)$.
- **Case 2:** $P$ is a point at infinity.

  Now, $l = 0$, so at least one of $h, k$ must be non-zero.

  If $h \neq 0$, view $C$ as the homogenization of $f_X(Y, Z) = f(1, Y, Z)$.
  $(k/h, l/h)$ is a finite point on $f_X$. Apply Case 1.

  If $k \neq 0$, view $C$ as the homogenization of $f_Y(X, Z) = f(X, 1, Z)$.
  $(h/k, l/k)$ is a finite point on $f_Y$. Apply Case 1.
- $C$ is a **smooth curve** if it is smooth at every rational point on it.

# Types of Singularity

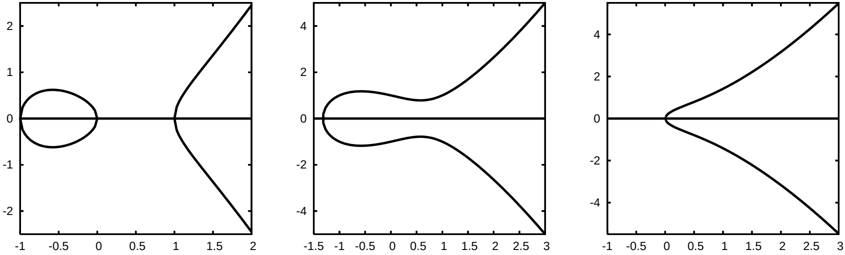



(a) (b) (c)

- (a) A **cusp** or a **spinode**: $Y^2 = X^3$.
- (b) A **loop** or a **double-point** or a **crunode**: $Y^2 = X^3 + X^2$.
- (c) An **isolated point** or an **acnode**: $Y^2 = X^3 - X^2$

- For a *real* curve $f(X, Y) = 0$, the type of singularity is determined by the matrix Hessian$(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$.

# Examples of Smooth Curves




(a) $Y^2 = X^3 - X$ (b) $Y^2 = X^3 - X + 1$ (c) $Y^2 = X^3 + X$

- An elliptic or hyperelliptic curve is needed to be smooth by definition.
- A curve of the form $Y^2 = v(X)$ is smooth if and only if $v(X)$ does not contain repeated roots.
- The point at infinity on an elliptic or hyperelliptic curve is never a point of singularity.

# Polynomial and Rational Functions on Curves

Let $C : f(X,Y) = 0$ be a curve defined by an *irreducible polynomial* $f(X,Y) \in K[X,Y]$.

- Let $G(X,Y), H(X,Y) \in K[X,Y]$ with $f|(G-H)$. Then, $G(P) = H(P)$ for every rational point $P$ on $C$ (since $f(P) = 0$). Thus, $G$ and $H$ represent the same function on $C$.
- Define $G(X,Y) \equiv H(X,Y) \pmod{f(X,Y)}$ if and only if $f|(G-H)$.
- Congruence modulo $f$ is an equivalence relation on $K[X,Y]$.
- Call the equivalence classes of $X$ and $Y$ by $x$ and $y$.
- The equivalence class of $G(X,Y)$ is $G(x,y)$.
- $K[C] = K[X,Y]/\langle f(X,Y)\rangle = K[x,y]$ is an integral domain.
- The field of fractions of $K[C]$ is $K(C) = \{G(x,y)/H(x,y) \mid H(x,y) \neq 0\} = K(x,y)$.

# Polynomial and Rational Functions on Elliptic and Hyperelliptic Curves

Consider the elliptic curve $Y^2 + u(X)Y = v(X)$, where $u(X) = a_1X + a_3$ and $v(X) = X^3 + a_2X^2 + a_4X + a_6$.

- $y^2 = -u(x)y + v(x)$.
- Every polynomial function on $C$ can be represented uniquely as $a(x) + yb(x)$ with $a(x), b(x) \in K[x]$.
- For $G(x,y) = a(x) + yb(x) \in K[C]$, define:
  - **Conjugate of $G$:** $\hat{G}(x,y) = a(x) - b(x)(u(x) + y)$.
  - **Norm of $G$:** $\mathrm{N}(G) = G\hat{G}$.
- $\mathrm{N}(G) = a(x)^2 - a(x)b(x)u(x) - v(x)b(x)^2 \in K[x]$.
- Every rational function on $C$ can be represented as $s(x) + yt(x)$ with $s(x), t(x) \in K(x)$.
- $K(C)$ is the quadratic extension of $K(X)$ obtained by adjoining a root of the irreducible polynomial $Y^2 + u(X)Y - v(X) \in K(X)[Y]$. The current notion of conjugacy coincides with the standard notion for field extensions.

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- These results hold equally well for hyperelliptic curves too.

# Poles and Zeros of Rational Functions

Let $C : f(X,Y) = 0$ be a plane (irreducible) curve, and $P = (h,k)$ a finite point on $C$.

- Let $G(x,y) \in K[C]$. The **value** of $G$ at $P$ is $G(P) = G(h,k) \in K$.
- A rational function $R(x,y) \in K(C)$ is **defined** at $P$ if there is a representation $R(x,y) = G(x,y)/H(x,y)$ for some polynomials $G, H$ with $H(P) = H(h,k) \neq 0$. In that case, the **value** of $R$ at $P$ is defined as $R(P) = G(P)/H(P) = G(h,k)/H(h,k) \in K$.
- If $R(x,y)$ is not defined at $P$, we take $R(P) = \infty$.
- Let $R(x,y) \in K(C)$ and $P$ a finite point on $C$.
  - $P$ is a **zero** of $R$ is $R(P) = 0$.
  - $P$ is a **pole** of $R$ is $R(P) = \infty$.
- The set of rational functions on $C$ defined at $P$ is a local ring with the unique maximal ideal comprising functions that evaluate to 0 at $P$.
- The notion of value of a rational function can be extended to the points at infinity on $C$.

# Value of a Rational Function at $\mathcal{O}$: Example

Let $C$ be an elliptic curve with $\mathcal{O}$ the point at infinity.

- Neglecting lower-degree terms gives $Y^2 \approx X^3$.
- $X$ is given a weight 2, and $Y$ a weight 3.
- Let $G(x, y) = a(x) + yb(x) \in K[C]$. Define the **degree** of $G$ as $\deg G = \max(2 \deg_x(a), 3 + 2 \deg_x(b))$.
- The **leading coefficient** of $G$ is that of $a$ or $b$ depending upon whether $2 \deg_x(a) > 3 + 2 \deg_x(b)$ or not.
- Let $R(x, y) = G(x, y)/H(x, y) \in K(C)$. Define $R(\mathcal{O})$ as:
  - 0 if $\deg G < \deg H$.
  - $\infty$ if $\deg G > \deg H$.
  - The ratio of the leading coefficients of $G$ and $H$, if $\deg G = \deg H$.

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- For hyperelliptic curves, analogous results hold. Now, $X$ and $Y$ are given weights 2 and $2g + 1$ respectively.

# Multiplicities of Poles and Zeros

Let $C$ be a curve, and $P$ a rational point on $C$.

- There exists a rational function $U_P(x, y)$ (depending on $P$) such that:
  1. $U_P(P) = 0$, and
  2. every rational function $R(x, y) \in K(C)$ can be expressed as $R = U_P^d S$ with $S$ having neither a pole nor a zero at $P$.
- $U_P$ is called a **uniformizer**.
- The integer $d$ is independent of the choice of $U_P$.
- Define the **order** of $R$ at $P$ as $\operatorname{ord}_P(R) = d$.
- $P$ is a **zero** of $R$ if and only if $\operatorname{ord}_P(R) > 0$. **Multiplicity** is $\operatorname{ord}_P(R)$.
- $P$ is a **pole** of $R$ if and only if $\operatorname{ord}_P(R) < 0$. **Multiplicity** is $-\operatorname{ord}_P(R)$.
- $P$ is neither a pole nor a zero of $R$ if and only if $\operatorname{ord}_P(R) = 0$.
- Any (non-zero) rational function has only finitely many poles and zeros.
- For a *projective* curve over an *algebraically closed* field, the sum of the orders of the poles and zeros of a (non-zero) rational function is 0.

# Poles and Zeros for Elliptic Curves

Let $C : Y^2 + u(X)Y = v(X)$ be an elliptic curve with $\mathcal{O}$ the point at infinity, and $P = (h, k)$ a finite point on $C$.

- The **opposite** of $P$ is defined as $\tilde{P} = (h, -k - u(h))$. $P$ and $\tilde{P}$ are the only points on $C$ with $X$-coordinate equal to $h$.
- The opposite of $\mathcal{O}$ is $\mathcal{O}$ itself.
- $P$ is called an **ordinary point** if $\tilde{P} \neq P$.
- $P$ is called a **special point** if $\tilde{P} = P$.
- Any line passing through $P$ but not a tangent to $C$ at $P$ can be taken as a **uniformizer** $U_P$ at $P$.
- For example, we may take $U_P = \begin{cases} x - h & \text{if } P \text{ is an ordinary point,} \\ y - k & \text{if } P \text{ is a special point.} \end{cases}$
- A **uniformizer** at $\mathcal{O}$ is $x/y$.

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- For hyperelliptic curves, identical results hold. A uniformizer at $\mathcal{O}$ is $x^g/y$.

# Multiplicities of Poles and Zeros for Elliptic Curves

- Let $G(x,y) = a(x) + yb(x) \in K[C]$.
- Let $e$ be the largest exponent for which $(x-h)^e$ divides both $a(x)$ and $b(x)$.
- Write $G(x,y) = (x-h)^e G_1(x,y)$.
- Take $l = 0$ if $G_1(h,k) \neq 0$.
- If $G_1(h,k) = 0$, take $l$ to be the largest exponent for which $(x-h)^l \mid \mathrm{N}(G_1)$.
- $\operatorname{ord}_P(G) = \begin{cases} e+l & \text{if } P \text{ is an ordinary point,} \\ 2e+l & \text{if } P \text{ is a special point.} \end{cases}$
- $\operatorname{ord}_{\mathcal{O}}(G) = -\max(2\deg_x a, 3 + 2\deg_x b)$.
- For a rational function $R(x,y) = G(x,y)/H(x,y) \in K(C)$, we have $\operatorname{ord}_P(R) = \operatorname{ord}_P(G) - \operatorname{ord}_P(H)$.

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- For hyperelliptic curves, identical results hold.
The order of $G$ at $\mathcal{O}$ is $\operatorname{ord}_{\mathcal{O}}(G) = -\max(2\deg_x a, 2g + 1 + 2\deg_x b)$.

# Poles and Zeros on Elliptic Curves: Examples

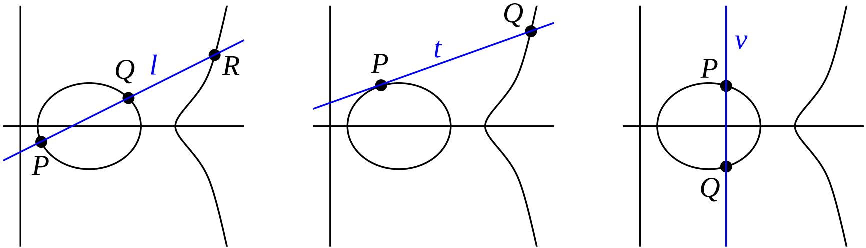
Consider the elliptic curve $C : Y^2 = X^3 - X$.

- Rational functions involving only $x$ are simpler. $R_1 = \frac{(x-1)(x+1)}{x^3(x-2)}$ has simple zeros at $x = \pm 1$, a simple pole at $x = 2$, and a pole of multiplicity three at $x = 0$. The points on $C$ with these $x$-coordinates are $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (-1, 0)$, $P_4 = (2, \sqrt{6})$ and $P_5 = (2, -\sqrt{6})$. $P_1, P_2, P_3$ are special points, so $\mathrm{ord}_{P_1}(R_1) = -6$, $\mathrm{ord}_{P_2}(R_1) = \mathrm{ord}_{P_3}(R_1) = 2$. $P_4$ and $P_5$ are ordinary points, so $\mathrm{ord}_{P_4}(R_1) = \mathrm{ord}_{P_5}(R_1) = -1$. Finally, note that $R_1 \to \frac{1}{x^2}$ as $x \to \infty$. But $x$ has a weight of 2, so $R_1$ has a zero of order 4 at $\mathcal{O}$. The sum of these orders is $-6 + 2 + 2 - 1 - 1 + 4 = 0$.

- Now, consider the rational function $R_2 = \frac{x}{y}$ involving $y$. At the point $P_1 = (0, 0)$, $R_2$ appears to be undefined. But $y^2 = x^3 - x$, so $R_2 = \frac{y}{x^2-1}$ too, and $R_2(P_1) = 0$, that is, $R_2$ has a zero at $P_1$. Using the explicit formula on $y$, show that $e = 0$ and $l = 1$. So $\mathrm{ord}_{P_1}(R_2) = 1$. On the other hand, the denominator $x^2 - 1$ has neither a pole nor a zero at $P_1$. So $\mathrm{ord}_{P_1}(R_2) = 1$.

- $\mathrm{ord}_{P_1}(x) = 2$ (since $e = 1$, $l = 0$, and $P_1$ is a special point), so the representation $R_2 = \frac{x}{y}$ also gives $\mathrm{ord}_{P_1}(R_2) = 2 - 1 = 1$.

# Poles and Zeros of a Line: Example




(a) (b) (c)

- (a) $\mathrm{ord}_P(l) = \mathrm{ord}_Q(l) = \mathrm{ord}_R(l) = 1$ and $\mathrm{ord}_{\mathcal{O}}(l) = -3$.
- (b) $\mathrm{ord}_P(t) = 2$, $\mathrm{ord}_Q(t) = 1$ and $\mathrm{ord}_{\mathcal{O}}(t) = -3$.
- (c) $\mathrm{ord}_P(v) = \mathrm{ord}_Q(v) = 1$ and $\mathrm{ord}_{\mathcal{O}}(v) = -2$.

# Formal Sums and Free Abelian Groups

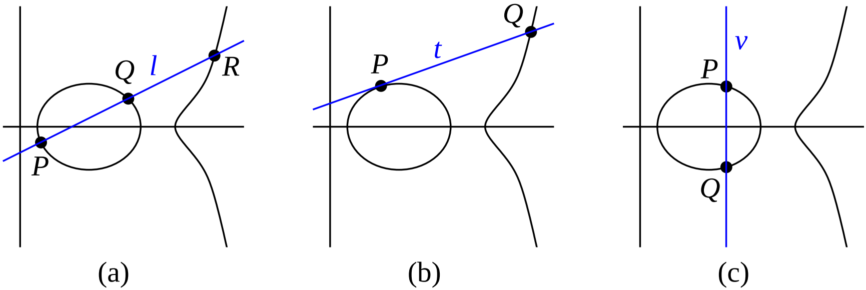
- Let $a_i$, $i \in I$, be *symbols* indexed by $I$.
- A **finite formal sum** of $a_i$, $i \in I$, is an expression of the form $\sum_{i \in I} m_i a_i$ with $m_i \in \mathbb{Z}$ such that $m_i = 0$ except for only finitely many $i \in I$.
- The sum $\sum_{i \in I} m_i a_i$ is formal in the sense that the symbols $a_i$ are not meant to be evaluated. They act as *placeholders*.
- Define $\sum_{i \in I} m_i a_i + \sum_{i \in I} n_i a_i = \sum_{i \in I} (m_i + n_i) a_i$
- Also define $-\sum_{i \in I} m_i a_i = \sum_{i \in I} (-m_i) a_i$
- The set of all finite formal sums is an Abelian group called the **free Abelian group** generated by $a_i$, $i \in I$.

# Divisors on Curves

Let $C$ be a projective curve defined over $K$.
$K$ is assumed to be *algebraically closed*.

- A **divisor** is a formal sum of the $K$-rational points on $C$.
- Notation: $D = \sum_P m_P[P]$.
- The **support** of $D$ is the set of points $P$ for which $m_P \neq 0$.
- The **degree** of $D$ is the sum $\sum_P m_P$.
- All divisors on $C$ form a group denoted by $\mathrm{Div}_K(C)$ or $\mathrm{Div}(C)$.
- All divisors on $C$ of degree 0 form a subgroup denoted by $\mathrm{Div}_K^0(C)$ or $\mathrm{Div}^0(C)$.
- **Divisor of a rational function** $R(x, y)$ is $\mathrm{Div}(R) = \sum_P \mathrm{ord}_P(R)[P]$.
- A **principal divisor** is the divisor of a rational function.
- Principal divisors satisfy: $\mathrm{Div}(R) + \mathrm{Div}(S) = \mathrm{Div}(RS)$ and $\mathrm{Div}(R) - \mathrm{Div}(S) = \mathrm{Div}(R/S)$.

# Divisor of a line: Example




- (a) $\mathrm{Div}(l) = [P] + [Q] + [R] - 3[\mathcal{O}]$.
- (b) $\mathrm{Div}(t) = 2[P] + [Q] - 3[\mathcal{O}]$.
- (c) $\mathrm{Div}(v) = [P] + [Q] - 2[\mathcal{O}]$.

# Picard Groups and Jacobians

- Suppose that $K$ is algebraically closed.
- Every principal divisor belongs to $\mathrm{Div}_K^0(C)$.
- The set of all principal divisors is a subgroup of $\mathrm{Div}_K^0(C)$, denoted by $\mathrm{Prin}_K(C)$ or $\mathrm{Prin}(C)$.
- Two divisors in $\mathrm{Div}_K(C)$ are called **equivalent** if they differ by the divisor of a rational function.
- The quotient group $\mathrm{Div}_K(C)/\mathrm{Prin}_K(C)$ is called the **divisor class group** or the **Picard group**, denoted $\mathrm{Pic}_K(C)$ or $\mathrm{Pic}(C)$.
- The quotient group $\mathrm{Div}_K^0(C)/\mathrm{Prin}_K(C)$ is called the **Jacobian** of $C$, denoted $\mathrm{Pic}_K^0(C)$ or $\mathrm{Pic}^0(C)$ or $\mathbb{J}_K(C)$ or $\mathbb{J}(C)$.
- If $K$ is not algebraically closed, $\mathbb{J}_K(C)$ is a particular subgroup of $\mathbb{J}_{\bar{K}}(C)$.

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- Elliptic- and hyperelliptic-curve cryptography deals with the Jacobian of elliptic and hyperelliptic curves.
- For elliptic curves, the Jacobian can be expressed by a more explicit **chord-and-tangent** rule.

# Divisors and the Chord-and-Tangent Rule

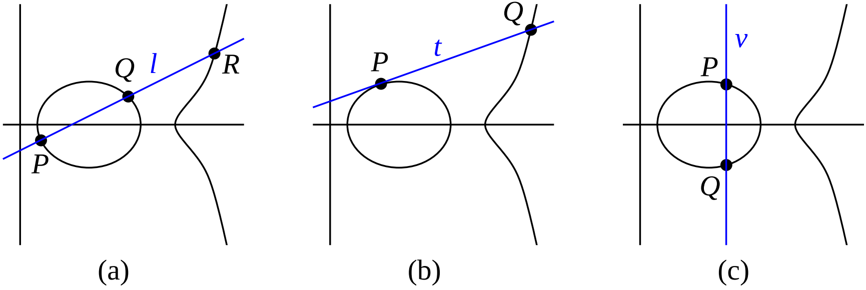
Let $C$ be an elliptic curve over an algebraically closed field $K$.

- For every $D \in \mathrm{Div}_K^0(C)$, there exist a unique rational point $P$ and a rational function $R$ such that $D = [P] - [\mathcal{O}] + \mathrm{Div}(R)$.
- $D$ is equivalent to $[P] - [\mathcal{O}]$ in $\mathbb{J}_K(C)$.
- Identify $P$ with the equivalence class of $[P] - [\mathcal{O}]$ in $\mathbb{J}_K(C)$.
- This identification yields a bijection between the set of rational points on $C$ and its Jacobian $\mathbb{J}_K(C)$.
- This bijection also leads to the chord-and-tangent rule in the following sense:

Let $D = \sum_P m_P[P] \in \mathrm{Div}_K(C)$. Then, $D$ is a principal divisor if and only if

- $\sum_P m_P = 0$ (integer sum), and
- $\sum_p m_P P = \mathcal{O}$ (sum under the chord-and-tangent rule).

# Illustrations of the Chord-and-Tangent Rule




- **Identity:** $\mathcal{O}$ is identified with $[\mathcal{O}] - [\mathcal{O}] = 0 = \mathrm{Div}(1)$.
- **Opposite:** By Part (c), $\mathrm{Div}(v) = ([P] - [\mathcal{O}]) + ([Q] - [\mathcal{O}])$ is 0 in $\mathbb{J}(C)$. By the correspondence, $P + Q = \mathcal{O}$, that is, $Q = -P$.
- **Sum:** By Part (a), $\mathrm{Div}(l) = ([P] - [\mathcal{O}]) + ([Q] - [\mathcal{O}]) + ([R] - [\mathcal{O}])$ is 0 in $\mathbb{J}(C)$, that is, $P + Q + R = \mathcal{O}$, that is, $P + Q = -R$.
- **Double:** By Part (b), $\mathrm{Div}(t) = ([P] - [\mathcal{O}]) + ([P] - [\mathcal{O}]) + ([Q] - [\mathcal{O}])$ is 0 in $\mathbb{J}(C)$, that is, $P + P + Q = \mathcal{O}$, that is, $2P = -Q$.

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# Part II

## Elliptic Curves

- Rational Maps and Endomorphisms on Elliptic Curves
- Multiplication-by-$m$ Maps and Division Polynomials
- Weil and Tate Pairing

# Notations and Assumptions

- $K$ is a field.
- $\bar{K}$ is the algebraic closure of $K$.
- Quite often, we will have $K = \mathbb{F}_q$ with $p = \operatorname{char} K$.
- $E : Y^2 + (a_1X + a_3)Y = X^3 + a_2X^2 + a_4X + a_6$ is an elliptic curve defined over $K$ (that is, $a_i \in K$).
- If $L$ is any field with $K \subseteq L \subseteq \bar{K}$, then $E$ is defined over $L$ as well.
- $E_L$ denotes the set of $L$-rational points on $E$.
- $E_L$ always contains the point $\mathcal{O}$ at infinity.
- If $L = \mathbb{F}_{q^k}$, we write $E_{q^k}$ as a shorthand for $E_L$.
- $E$ (without any subscript) means $E_{\bar{K}}$.
- A rational function $R$ on $E$ is an element of $\bar{K}(E)$.
- $R$ is defined over $L$ if $R$ has a representation $R = G(x,y)/H(x,y)$ with $G, H \in L[x,y]$.

# Elliptic Curves Over Finite Fields

- Let $K$ be not algebraically closed (like $K = \mathbb{F}_q$).
- The group $E_{\bar{K}}$ is isomorphic to $\mathbb{J}_{\bar{K}}(E)$.
- The one-to-one correspondence of $\mathbb{J}_{\bar{K}}(E)$ with $E_{\bar{K}}$ allows us to use the chord-and-tangent rule.
- If $P$ and $Q$ are $K$-rational, then the chord-and-tangent rule guarantees that $P + Q$ is $K$-rational too.
- All $K$-rational points in $E_{\bar{K}}$ together with $\mathcal{O}$ constitute a subgroup of $E_{\bar{K}}$.
- Denote this subgroup by $E_K$.
- $E_K$ can be identified with a subgroup $\mathbb{J}_K(E)$ of $\mathbb{J}_{\bar{K}}(E)$.
- Since $K$ is not algebraically closed, $\mathbb{J}_K(E)$ cannot be defined like $\mathbb{J}_{\bar{K}}(E)$.
- Thanks to the chord-and-tangent rule, we do not need to worry too much about $\mathbb{J}_K(E)$ (at least so long as computational issues are of only concern).

# Discriminants and *j*-invariants

- Define the following quantities for $E$:

$$\begin{aligned} d_2 &= a_1^2 + 4a_2 \\ d_4 &= 2a_4 + a_1a_3 \\ d_6 &= a_3^2 + 4a_6 \\ d_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\ c_4 &= d_2^2 - 24d_4 \\ \Delta(E) &= -d_2^2d_8 - 8d_4^3 - 27d_6^2 + 9d_2d_4d_6 \\ j(E) &= c_4^3/\Delta(E), \ \text{ if } \Delta(E) \neq 0\,. \end{aligned}$$

- $\Delta(E)$ is called the **discriminant** of $E$.
- $j(E)$ is called the ***j*-invariant** of $E$.
- $E$ is smooth (that is, an elliptic curve) if and only if $\Delta(E) \neq 0$.
- $j(E)$ is defined for every elliptic curve.
- For two elliptic curves $E, E'$, we have $j(E) = j(E')$ if and only if $E$ and $E'$ are isomorphic.

# Addition Formula for the General Weierstrass Equation

Let $P = (h_1, k_1)$ and $Q = (h_2, k_2)$ be points on $E.$ Assume that $P, Q, P + Q$ are not $\mathcal{O}$. Let $R = (h_3, k_3) = P + Q$.

$$h_3 = \lambda^2 + a_1\lambda - a_2 - h_1 - h_2, \text{ and}$$

$$k_3 = -(\lambda + a_1)h_3 - \mu - a_3, \text{ where}$$

$$\lambda = \begin{cases} \dfrac{k_2 - k_1}{h_2 - h_1} & \text{if } P \neq Q, \\ \\ \dfrac{3h_1^2 + 2a_2h_1 + a_4 - a_1k_1}{2k_1 + a_1h_1 + a_3} & \text{if } P = Q, \text{ and} \end{cases}$$

$$\mu = k_1 - \lambda h_1 \ .$$

The opposite of $(h, k)$ is $(h, -k - a_1h - a_3)$.

# Choosing a Random Point on an Elliptic Curve

Let $E : Y^2 + (a_1X + a_3)Y = X^3 + a_2X^2 + a_4X + a_6$ be defined over $K$. To obtain a random point $P = (h, k) \in E_K$.

- Choose the $X$-coordinate $h$ randomly from $K$.
- The corresponding $Y$-coordinates are roots of

$$Y^2 + (a_1h + a_3)Y - (h^3 + a_2h^2 + a_4h + a_6).$$

- This polynomial is either irreducible over $K$ or has two roots in $K$.
- If $K$ is algebraically closed, then this polynomial has roots in $K$.
- If $K$ is a finite field, then, with probability about $1/2$, this polynomial has roots in $K$.
- Use a root-finding algorithm to compute a root $k$.
- Output $(h, k)$.

# Rational Maps on Elliptic Curves

- A **rational map** on $E$ is a function $E \to E$.
- A rational map $\alpha$ is specified by two rational functions $\alpha_1, \alpha_2 \in \bar{K}(E)$ such that, for any point $P \in E$, $\alpha(P) = \alpha(h,k) = (\alpha_1(h,k), \alpha_2(h,k))$ is again a point on $E$.
- Since $\alpha(P)$ is a point on $E$, $\alpha_1, \alpha_2$ satisfy the equation for $E$ and constitute the elliptic curve $E_{\bar{K}(E)}$.
- Denote the point at infinity on this curve by $\mathcal{O}'$. Define $\mathcal{O}'(P) = \mathcal{O}$ for all $P \in E$.
- For a non-zero $\alpha \in E_{\bar{K}(E)}$ and a point $P \in E$, either both $\alpha_1(P), \alpha_2(P)$ are defined at $P$, or both are undefined at $P$. In the first case, we take $\alpha(P) = (\alpha_1(P), \alpha_2(P))$, and in the second case, $\alpha(P) = \mathcal{O}$.
- The addition of $E_{\bar{K}(E)}$ is compatible with the addition of $E$, that is, $(\alpha + \beta)(P) = \alpha(P) + \beta(P)$ for all $\alpha, \beta \in E_{\bar{K}(E)}$ and $P \in E$.
- A rational map is either constant or surjective.

# Rational Maps: Examples

- The **zero map** $\mathcal{O}' : E \to E$, $P \mapsto \mathcal{O}$.
- The **identity map** $\mathrm{id} : E \to E$, $P \mapsto P$.
- The **translation map** $\tau_Q : E \to E$, $P \mapsto P + Q$, for a fixed $Q \in E$.
- The **multiplication-by-*m* map** $[m] : E \to E$, $P \mapsto mP$, where $m \in \mathbb{Z}$.
- The **Frobenius map** $\varphi$**:**
  - $E$ is defined over $K = \mathbb{F}_q$.
  - For $a \in \bar{K}$, $a^q = a$ if and only if $a \in \mathbb{F}_q$.
  - For $P = (h, k) \in E$, the point $(h^q, k^q) \in E$.
  - Define $\varphi(h, k) = (h^q, k^q)$.

# Endomorphisms

- A rational map on $E$, which is also a group homomorphism of $E$, is called an **endomorphism** or an **isogeny**.
- The set of all endomorphisms of $E$ is denoted by $\mathrm{End}(E)$.
- Define addition in $\mathrm{End}(E)$ as $(\alpha + \beta)(P) = \alpha(P) + \beta(P)$.
- Define multiplication in $\mathrm{End}(E)$ as $(\alpha \circ \beta)(P) = \alpha(\beta(P))$.
- $\mathrm{End}(E)$ is a ring under these operations. The additive identity is $\mathcal{O}'$. The multiplicative identity is id.
- All multiplication-by-$m$ maps $[m]$ are endomorphisms. We have $[m] \neq [n]$ for $m \neq n$.
- The translation map $\tau_Q$ is not an endomorphism unless $Q = \mathcal{O}$.
- The Frobenius map $\varphi$ is an endomorphism with $\varphi \neq [m]$ for any $m$.
- If $\mathrm{End}(E)$ contains a map other than the maps $[m]$, $E$ is called a curve with **complex multiplication**.

# The Multiplication-by-*m* Maps

- Identify $[m]$ as a pair $(g_m, h_m)$ of rational functions.
- $g_1 = x$, $h_1 = y$.
- $g_2 = -2x + \lambda^2 + a_1\lambda - a_2$ and
$h_2 = -\lambda(g_2 - x) - a_1 g_2 - a_3 - y$,
where $\lambda = \dfrac{3x^2 + 2a_2x + a_4 - a_1y}{2y + a_1x + a_3}$.
- For $m \geqslant 3$, we have the recursive definition:
$$g_m = -g_{m-1} - x + \lambda^2 + a_1\lambda - a_2 \text{ and}$$
$$h_m = -\lambda(g_m - x) - a_1 g_m - a_3 - y,$$
where $\lambda = \dfrac{h_{m-1} - y}{g_{m-1} - x}$.

# The Group of $m$-torsion Points

- For $m \in \mathbb{N}$, define $E[m] = \{P \in E \mid mP = \mathcal{O}\}$.
- Recall that $p = \operatorname{char} K$.
- If $p = 0$ or $\gcd(p, m) = 1$, then $E[m] \cong \mathbb{Z}_m \times \mathbb{Z}_m$, and so $|E[m]| = m^2$.
- Suppose that $p > 0$. Let $m = p^\nu m'$ with $\gcd(m', p) = 1$. Then,
$$E[m] \cong \begin{cases} \mathbb{Z}_{m'} \times \mathbb{Z}_{m'} & \text{if } E[p] = \{\mathcal{O}\}, \\ \mathbb{Z}_{m'} \times \mathbb{Z}_m & \text{otherwise.} \end{cases}$$
- If $\gcd(m, n) = 1$, we have $E[mn] \cong E[m] \times E[n]$.

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- For a subset $S \subseteq E$, define the divisor $[S] = \sum_{P \in S}[P]$.
- If $p \neq 2, 3$ and $m, n, m+n, m-n$ are all coprime to $p$, we have $\operatorname{Div}(g_m - g_n) = [E[m+n]] + [E[m-n]] - 2[E[m]] - 2[E[n]]$.
- If $p \in \{2, 3\}$, $\gcd(m, p) = 1$, and $n = p^\nu n'$ with $\nu \geqslant 1$ and $\gcd(n', p) = 1$, we have $\operatorname{Div}(g_m - g_n) = [E[m+n]] + [E[m-n]] - 2[E[m]] - 2\alpha^\nu[E[n]]$.

# Division Polynomials

- The rational functions $g_m, h_m$ have poles precisely at the points in $E[m]$. But they have some zeros also.
- We investigate polynomials having zeros precisely at the points of $E[m]$.
- Assume that either $p = 0$ or $\gcd(p, m) = 1$.
- $E[m]$ contains exactly $m^2$ points with $\sum_{P \in E[m]} P = \mathcal{O}$.
- Consider the degree-zero divisor $[E[m]] - m^2[\mathcal{O}] = \sum_{P \in E[m]} [P] - m^2[\mathcal{O}]$.
- There exists a rational function $\psi_m$ with $\mathrm{Div}(\psi_m) = [E[m]] - m^2[\mathcal{O}]$.
- Since the only pole of $\psi_m$ is at $\mathcal{O}$, $\psi_m$ is a polynomial function.
- $\psi_m$ is unique up to multiplication of elements of $\bar{K}^*$.
- If we arrange the leading coefficient of $\psi_m$ to be $m$, then $\psi_m$ becomes unique and is called the ***m*-th division polynomial**.

# Division Polynomials: Explicit Formulas

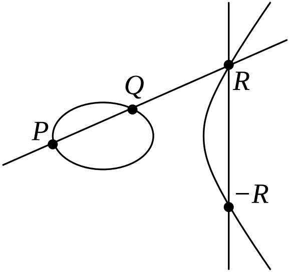
- $$\begin{aligned} \psi_0 &= 0 \\ \psi_1 &= 1 \\ \psi_2 &= 2y + a_1x + a_3 \\ \psi_3 &= 3x^4 + d_2x^3 + 3d_4x^2 + 3d_6x + d_8 \\ \psi_4 &= \left[2x^6 + d_2x^5 + 5d_4x^4 + 10d_6x^3 + 10d_8x^2 + \right. \\ &\qquad \left. (d_2d_8 - d_4d_6)x + d_4d_8 - d_6^2\right] \psi_2 \\ \psi_{2m} &= \frac{(\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2)\psi_m}{\psi_2} \text{ for } m > 2 \\ \psi_{2m+1} &= \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \text{ for } m \geqslant 2. \end{aligned}$$
- $g_m - g_n = -\dfrac{\psi_{m+n}\psi_{m-n}}{\psi_m^2\psi_n^2}$. Putting $n = 1$ gives $g_m = x - \dfrac{\psi_{m+1}\psi_{m-1}}{\psi_m^2}$.
- $h_m = \dfrac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{2\psi_2\psi_m^3} - \dfrac{1}{2}(a_1g_m + a_3)$

$$= y + \frac{\psi_{m+2}\psi_{m-1}^2}{\psi_2\psi_m^3} + (3x^2 + 2a_2x + a_4 - a_1y)\frac{\psi_{m-1}\psi_{m+1}}{\psi_2\psi_m^2}.$$

# Size and Structure of $E_q$

- **Hasse's Theorem:** $|E_q| = q + 1 - t$ with $-2\sqrt{q} \leqslant t \leqslant 2\sqrt{q}$.
- $t$ is called the **trace of Frobenius** at $q$.
- The Frobenius endomorphism satisfies $\varphi \circ \varphi - [t] \circ \varphi + [q] = \mathcal{O}'$.
- Let $L = \mathbb{F}_{q^k}$ be an extension of $K = \mathbb{F}_q$.
- Let $W^2 - tW + q = (W - \alpha)(W - \beta)$ with $\alpha, \beta \in \mathbb{C}$.
- **Weil's Theorem:** $|E_{q^k}| = q^k + 1 - (\alpha^k + \beta^k)$.
- **Example:** Consider $E : Y^2 = X^3 + X + 1$ defined over $\mathbb{F}_5$. $E_5$ contains the nine points $\mathcal{O}$, $(0, \pm 1)$, $(2, \pm 1)$, $(3, \pm 1)$ and $(4, \pm 2)$, so that $|E_5| = 9 = (5 + 1) - t$, that is, $t = -3$.
  Consider $(W - \alpha)(W - \beta) = W^2 - tW + q = W^2 + 3W + 5$, that is, $\alpha + \beta = -3$ and $\alpha\beta = 5$. But then $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 10 = -1$. Therefore, $|E_{25}| = 25 + 1 - (-1) = 27$.
- **Structure Theorem for $E_q$:**
  $E_q$ is either cyclic or isomorphic to $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ with $n_1, n_2 \geqslant 2$, $n_1 | n_2$, and $n_1 | (q - 1)$.

# More on Divisors




- $\mathrm{Div}(L_{P,Q}) = [P] + [Q] + [R] - 3[\mathcal{O}]$.
- $\mathrm{Div}(L_{R,-R}) = [R] + [-R] - 2[\mathcal{O}]$.
- $\mathrm{Div}(L_{P,Q}/L_{R,-R}) = [P] + [Q] - [-R] - [\mathcal{O}] = [P] + [Q] - [P+Q] - [\mathcal{O}]$.
- $[P] - [\mathcal{O}]$ is equivalent to $[P+Q] - [Q]$.
- $([P] - [\mathcal{O}]) + ([Q] - [\mathcal{O}])$ is equivalent to $[P+Q] - [\mathcal{O}]$.
- For both these cases of equivalence, the pertinent rational function is $L_{P,Q}/L_{P+Q,-(P+Q)}$ which can be easily computed. We can force this rational function to have leading coefficient 1.

# More on Divisors (contd)

- Let $D = \sum_P n_P[P]$ be divisor on $E$ and $f \in \bar{K}(E)$ a rational function such that the supports of $D$ and $\mathrm{Div}(f)$ are disjoint. Define

$$f(D) = \prod_{P \in E} f(P)^{n_P} = \prod_{P \in \mathrm{Supp}(D)} f(P)^{n_P}.$$

- $\mathrm{Div}(f) = \mathrm{Div}(g)$ if and only if $f = cg$ for some non-zero constant $c \in \bar{K}^*$.
- If $D$ has degree 0, then $f(D) = g(D) \prod_P c^{n_P} = g(D) c^{\sum_P n_P} = g(D) c^0 = g(D)$.
- **Weil reciprocity theorem:** If $f$ and $g$ are two non-zero rational functions on $E$ such that $\mathrm{Div}(f)$ and $\mathrm{Div}(g)$ have disjoint supports, then

$$f(\mathrm{Div}(g)) = g(\mathrm{Div}(f)).$$

# Weil Pairing: Definition

Let $E$ be an elliptic curve defined over a finite field $K = \mathbb{F}_q$.
Take a positive integer $m$ coprime to $p = \operatorname{char} K$.
Let $\mu_m$ denote the $m$-th roots of unity in $\bar{K}$.
We have $\mu_m \subseteq \mathbb{F}_{q^k}$, where $k = \operatorname{ord}_m(q)$ is called the **embedding degree**.
Let $E[m]$ be those points in $E = E_{\bar{K}}$, whose orders divide $m$.

- **Weil pairing** is a function

$$e_m : E[m] \times E[m] \to \mu_m$$

defined as follows.

- Take $P_1, P_2 \in E[m]$.
- Let $D_1$ be a divisor equivalent to $[P_1] - [\mathcal{O}]$. Since $mP_1 = \mathcal{O}$, there exists a rational function $f_1$ such that $\operatorname{Div}(f_1) = mD_1 = m[P_1] - m[\mathcal{O}]$.
- Similarly, let $D_2$ be a divisor equivalent to $[P_2] - [\mathcal{O}]$. There exists a rational function $f_2$ such that $\operatorname{Div}(f_2) = mD_2 = m[P_2] - m[\mathcal{O}]$.
- $D_1$ and $D_2$ are chosen to have disjoint supports.
- Define $e_m(P_1, P_2) = f_1(D_2)/f_2(D_1)$.

# Weil Pairing is Well-defined

- $f_1$ and $f_2$ are unique up to multiplication by non-zero elements of $\bar{K}^*$. So $f_1(D_2)$ and $f_2(D_1)$ are independent of the choices of $f_1$ and $f_2$.
- Let $D_1' = D_1 + \operatorname{Div}(g)$ have disjoint support from $D_2$. But then $mD_1' = mD_1 + m\operatorname{Div}(g) = \operatorname{Div}(f_1) + \operatorname{Div}(g^m) = \operatorname{Div}(f_1 g^m)$. Therefore,

$$f_1 g^m(D_2)/f_2(D_1 + \operatorname{Div}(g)) = \frac{f_1(D_2)g^m(D_2)}{f_2(D_1)f_2(\operatorname{Div}(g))}$$
$$= \frac{f_1(D_2)g(mD_2)}{f_2(D_1)f_2(\operatorname{Div}(g))} = \frac{f_1(D_2)g(\operatorname{Div}(f_2))}{f_2(D_1)f_2(\operatorname{Div}(g))} = \frac{f_1(D_2)g(\operatorname{Div}(f_2))}{f_2(D_1)g(\operatorname{Div}(f_2))} = \frac{f_1(D_2)}{f_2(D_1)}.$$

  So $e_m(P_1, P_2)$ is independent of the choice of $D_1$ and likewise of $D_2$ too.
- It is customary to choose $D_2 = [P_2] - [\mathcal{O}]$ and $D_1 = [P_1 + T] - [T]$ for a point $T$ different from $-P_1$, $P_2$, $P_2 - P_1$, and $\mathcal{O}$. $T$ need not be in $E[m]$. One can take $T$ randomly from $E$.
- $e_m(P_1, P_2)^m = f_1(mD_2)/f_2(mD_1) = f_1(\operatorname{Div}(f_2))/f_2(\operatorname{Div}(f_1)) = 1$ (by Weil reciprocity), that is, $e_m(P_1, P_2)$ is indeed an $m$-th root of unity.

# Properties of Weil Pairing

Let $P, Q, R$ be arbitrary points in $E[m]$.

- **Bilinearity:**

$$\begin{aligned} e_m(P+Q,R) &= e_m(P,R)e_m(Q,R), \\ e_m(P,Q+R) &= e_m(P,Q)e_m(P,R). \end{aligned}$$

- **Alternating:** $e_m(P,P) = 1$.
- **Skew symmetry:** $e_m(Q,P) = e_m(P,Q)^{-1}$.
- **Non-degeneracy:** If $P \neq \mathcal{O}$, then $e_m(P,Q) \neq 1$ for some $Q \in E[m]$.
- **Compatibility:** If $S \in E[mn]$ and $Q \in E[n]$, then $e_{mn}(S,Q) = e_n(mS,Q)$.
- If $m$ is a prime and $P \neq \mathcal{O}$, then $e_m(P,Q) = 1$ if and only if $Q$ lies in the subgroup generated by $P$ (that is, $Q = aP$ for some integer $a$).

# Computing Weil Pairing: The Functions $f_{n,P}$

- Let $P \in E$.
- For $n \in \mathbb{Z}$, define the rational functions $f_{n,P}$ as having the divisor

$$\mathrm{Div}(f_{n,P}) = n[P] - [nP] - (n-1)[\mathcal{O}].$$

$f_{n,P}$ are unique up to multiplication by elements of $\bar{K}^*$.
We may choose the unique monic polynomial for $f_{n,P}$.

- $f_{n,P}$ satisfy the recurrence relation:

$$\begin{aligned} f_{0,P} &= f_{1,P} = 1, \\ f_{n+1,P} &= \left(\frac{L_{P,nP}}{L_{(n+1)P,-(n+1)P}}\right) f_{n,P} \text{ for } n \geqslant 1, \\ f_{-n,P} &= \frac{1}{f_{n,P}} \text{ for } n \geqslant 1. \end{aligned}$$

- If $P \in E[m]$, then $\mathrm{Div}(f_{m,P}) = m[P] - [mP] - (m-1)[\mathcal{O}] = m[P] - m[\mathcal{O}]$.
- Computing $f_{m,P}$ using the above recursive formula is too inefficient.

# Computing Weil Pairing: More about $f_{n,P}$

- The rational functions $f_{n,P}$ also satisfy

$$f_{n+n',P} = f_{n,P}\, f_{n',P} \times \left( \frac{L_{nP,n'P}}{L_{(n+n')P,-(n+n')P}} \right).$$

- In particular, for $n = n'$, we have

$$f_{2n,P} = f_{n,P}^2 \times \left( \frac{L_{nP,nP}}{L_{2nP,-2nP}} \right).$$

  Here, $L_{nP,nP}$ is the line tangent to $E$ at the point $nP$.

- This and the recursive expression of $f_{n+1,P}$ in terms of $f_{n,P}$ yield a repeated double-and-add algorithm.
- The function $f_{n,P}$ is usually kept in the factored form.
- It is often not necessary to compute $f_{n,P}$ explicitly. The value of $f_{n,P}$ at some point $Q$ is only needed.

# Miller's Algorithm for Computing $f_{n,P}$

- **Input:** A point $P \in E$ and a positive integer $n$.
- **Output:** The rational function $f_{n,P}$.
- **Steps**
  - Let $n = (n_s n_{s-1} \ldots n_1 n_0)_2$ be the binary representation of $n$ with $n_s = 1$.
  - Initialize $f = 1$ and $U = P$.
  - For $i = s-1, s-2, \ldots, 1, 0$, do the following:
    - /* Doubling */
      Update $f = f^2 \times \left(\frac{L_{U,U}}{L_{2U,-2U}}\right)$ and $U = 2U$.
    - /* Conditional adding */
      If $(n_i = 1)$, update $f = f \times \left(\frac{L_{U,P}}{L_{U+P,-(U+P)}}\right)$ and $U = U + P$.
  - Return $f$.
- **Note:** One may supply a point $Q \in E$ and wish to compute the value $f_{n,P}(Q)$ (instead of the function $f_{n,P}$). In that case, the functions $L_{U,U}/L_{2U,-2U}$ and $L_{U,P}/L_{U+P,-(U+P)}$ should be evaluated at $Q$ before multiplication with $f$.

# Weil Pairing and the Functions $f_{n,P}$

Let $P_1, P_2 \in E[m]$, and we want to compute $e_m(P_1, P_2)$.

- Choose a point $T$ not equal to $\pm P_1, -P_2, P_2 - P_1, \mathcal{O}$.
- We have $e_m(P_1, P_2) = \dfrac{f_{m,P_2}(T)\ \ f_{m,P_1}(P_2 - T)}{f_{m,P_1}(-T)\ \ f_{m,P_2}(P_1 + T)}$ .
- If $P_1 \neq P_2$, then we also have $e_m(P_1, P_2) = (-1)^m\ \dfrac{f_{m,P_1}(P_2)}{f_{m,P_2}(P_1)}$ .
- Miller's algorithm for computing $f_{n,P}(Q)$ can be used.
- All these invocations of Miller's algorithm have $n = m$.
- So a single double-and-add loop suffices.
- For efficiency, one may avoid the division operations in Miller's loop by separately maintaining polynomial expressions for the numerator and the denominator of $f$. After the loop terminates, a single division is made.

# Tate Pairing

Let $E$ be an elliptic curve defined over $K = \mathbb{F}_q$ with $p = \operatorname{char} K$.
Let $m$ be a positive integer coprime to $p$.
Let $k = \operatorname{ord}_m(q)$ (the **embedding degree**), and $L = \mathbb{F}_{q^k}$.
Let $E_L[m] = \{P \in E_L \mid mP = \mathcal{O}\}$, and $mE_L = \{mP \mid P \in E_L\}$.
Let $(L^*)^m = \{a^m \mid a \in L^*\}$ be the set of $m$-th powers in $L^*$.

- Let $P$ be a point in $E_L[m]$, and $Q$ a point in $E_L$.
- Since $mP = \mathcal{O}$, there is a rational function $f$ with $\operatorname{Div}(f) = m[P] - m[\mathcal{O}]$.
- Let $D$ be any divisor equivalent to $[Q] - [\mathcal{O}]$ with disjoint support from $\operatorname{Div}(f)$. It is customary to choose a point $T$ different from $-P, Q, Q-P, \mathcal{O}$ and take $D = [Q+T] - [T]$.
- The **Tate pairing** $\langle\ ,\ \rangle_m : E_L[m] \times E_L/mE_L \to L^*/(L^*)^m$ of $P$ and $Q$ is
$$\langle P, Q\rangle_m = f(D).$$
- $Q$ should be regarded as a point in $E_L/mE_L$.
- The value of $\langle P, Q\rangle_m$ is unique up to multiplication by an $m$-th power of a non-zero element of $L$, that is, $\langle P, Q\rangle_m$ is unique in $L^*/(L^*)^m$.

# Properties of Tate Pairing

- **Bilinearity:**

$$\begin{aligned}\langle P+Q,R\rangle_m &= \langle P,R\rangle_m\langle Q,R\rangle_m,\\ \langle P,Q+R\rangle_m &= \langle P,Q\rangle_m\langle P,R\rangle_m.\end{aligned}$$

- **Non-degeneracy:** For every $P \in E_L[m]$, $P \neq \mathcal{O}$, there exists $Q$ with $\langle P,Q\rangle_m \neq 1$. For every $Q \notin mE_L$, there exists $P \in E_L[m]$ with $\langle P,Q\rangle_m \neq 1$.
- The Weil pairing is related to the Tate pairing as

$$e_m(P,Q) = \frac{\langle P,Q\rangle_m}{\langle Q,P\rangle_m}$$

up to $m$-th powers.

- Let $k = \mathrm{ord}_m(q)$ be the embedding degree. The Tate pairing can be made unique by exponentiation to the power $(q^k-1)/m$:

$$\hat{e}_m(P,Q) = (\langle P,Q\rangle_m)^{\frac{q^k-1}{m}}$$

$\hat{e}_m(P,Q)$ is called the **reduced Tate pairing**. The reduced pairing continues to exhibit bilinearity and non-degeneracy.

# Computing the Tate Pairing

- Take $D = [Q+T] - [T]$, where $T \neq P, -Q, P-Q, \mathcal{O}$.
- We have $\langle P, Q \rangle_m = \dfrac{f_{m,P}(Q+T)}{f_{m,P}(T)}$ .
- Miller's algorithm is used to compute $\langle P, Q \rangle_m$.
- A single double-and-add loop suffices.
- For efficiency, the numerator and the denominator in $f$ may be updated separately. After the loop, a single division is made.
- If the reduced pairing is desired, then a final exponentiation to the power $(q^k - 1)/m$ is made on the value returned by Miller's algorithm.

# Weil vs. Tate Pairing

- The Miller loop for Tate pairing is more efficient than that for Weil pairing.
- The reduced Tate pairing demands an extra exponentiation.
- Let $k = \mathrm{ord}_m(q)$ be the embedding degree, and $L = \mathbb{F}_{q^k}$.
- Tate pairing requires working in the field $L$.
- Let $L'$ be the field obtained by adjoining to $L$ all the coordinates of $E[m] = E_{\bar{K}}[m]$.
- Weil pairing requires working in the field $L'$.
- $L'$ is potentially much larger than $L$.
- **Special case:** $m$ is a prime divisor of $|E_K|$ with $m \nmid q$ and $m \nmid (q-1)$. Then, $L' = L$. So it suffices to work in the field $L$ only.
- For cryptographic applications, Tate pairing is used more often that Weil pairing.
- One takes $\mathbb{F}_q$ with $|q|$ about 160–300 bits and $k \leqslant 12$. Larger embedding degrees are impractical for implementation.

# Distortion Maps

Let $m$ be a prime divisor of $|E_K|$.
Let $P$ be a generator of a subgroup $G$ of $E_K$ of order $m$.
**Goal:** To define a pairing of the points in $G$.

- If $k = 1$ (that is, $L = K$), then $\langle P, P \rangle_m \neq 1$.
- **Bad news:** If $k > 1$, then $\langle P, P \rangle_m = 1$.
  But then, by bilinearity, $\langle Q, Q' \rangle_m = 1$ for all $Q, Q' \in G$.
- **A way out:** If $k > 1$ and $Q \in L$ is linearly independent of $P$ (that is, $Q \notin G$), then $\langle P, Q \rangle_m \neq 1$.
- Let $\phi : E_L \to E_L$ be an endomorphism of $E_L$ with $\phi(P) \notin G$.
  $\phi$ is called a **distortion map**.
- Define the **distorted Tate pairing** of $P, Q \in G$ as $\langle P, \phi(Q) \rangle_m$.
- Since $\phi(P)$ is linearly independent of $P$, we have $\langle P, \phi(P) \rangle_m \neq 1$.
- Since $\phi$ is an endomorphism, bilinearity is preserved.
- **Symmetry:** We have $\langle Q, \phi(Q') \rangle_m = \langle Q', \phi(Q) \rangle_m$ for all $Q, Q' \in G$.
- Distortion maps exist only for supersingular curves.

# Twists

Let $E$ be defined by the short Weierstrass equation $Y^2 = X^3 + aX + b$. Let $d \geqslant 2$, and $v \in \mathbb{F}_q^*$ a $d$-th power non-residue.

- Consider the curve $E' : Y^2 = X^3 + v^{4/d}aX + v^{6/d}b$ (defined over $\mathbb{F}_{q^d}$).
- If $d = 2$, then $E'$ is defined over $\mathbb{F}_q$ itself.
- $E'$ is called a **twist of $E$ of degree $d$**.
- $E$ and $E'$ are isomorphic over $\mathbb{F}_{q^d}$. An explicit isomorphism is given by the map $\phi_d : E' \to E$ taking $(h, k) \mapsto (v^{-2/d}h, v^{-3/d}k)$.
- Let $m$ be a prime divisor of $|E_q|$, $G$ a subgroup of order $m$ in $E_{q^k}$, and $G'$ a subgroup of order $m$ in $E'_{q^k}$. Let $P, P'$ be generators of $G$ and $G'$. Suppose that $\phi_d(P')$ is linearly independent of $P$.
- For $d = 2$ (**quadratic twist**), a natural choice is $G \subseteq E_q$ and $G' \subseteq E'_q$.
- Define a pairing of points $Q \in G$ and $Q' \in G'$ as $\langle Q, \phi_d(Q') \rangle_m$.
- This is called the **twisted Tate pairing**.

# Pairing-friendly Curves

- **Requirement for efficient computation:** Small embedding degree $k$.
- For general curves, $k$ is quite high ($|k| \approx |m|$).
- Only some specific types of curves qualify as pairing-friendly.
- **Supersingular curves**
  - By Hasse's Theorem, $|E_q| = q + 1 - t$ with $|t| \leqslant 2\sqrt{q}$.
  - If $p|t$, we call $E$ a **supersingular curve**.
  - Curves of the form $Y^2 + aY = X^3 + bX + c$ are supersingular over fields of characteristic 2.
  - All supersingular curves over a finite field $K$ of characteristic 2 have $j$-invariant equal to 0, and so are isomorphic over $\bar{K}$. The same result holds for $p = 3$.
  - Supersingular curves have small embedding degrees. The only possibilities are $1, 2, 3, 4, 6$.
  - If $\mathbb{F}_q$ is a prime field with $q \geqslant 5$, the only possibility is $k = 2$.
  - Non-supersingular curves are called **ordinary curves**.
  - It is difficult to locate ordinary curves with small embedding degrees.

# How to Find Pairing-friendly Curves

- Let $k$ be a positive integer, and $\Delta$ a small positive square-free integer.
- Search for integer-valued polynomials $t(x), m(x), q(x) \in \mathbb{Q}[x]$ to represent a family of elliptic curves of embedding degree $k$ and discriminant $\Delta$. The triple $(t, m, q)$ should satisfy the following:
  1. $q(x) = p(x)^n$ for some $n \in \mathbb{N}$ and $p(x) \in \mathbb{Q}[x]$ representing primes.
  2. $m(x)$ is irreducible with a positive leading coefficient.
  3. $m(x) | q(x) + 1 - t(x)$.
  4. $m(x) | \Phi_k(t(x) - 1)$, where $\Phi_k$ is the $k$-th cyclotomic polynomial.
  5. There are infinitely many integers $(x, y)$ satisfying $\Delta y^2 = 4q(x) - t(x)^2$.
- If $y$ in Condition 5 can be parameterized by a polynomial $y(x) \in \mathbb{Q}[x]$, the family is called **complete**, otherwise it is called **sparse**.
- For obtaining ordinary curves, we require $\gcd(q(x), m(x)) = 1$.
- The **complex multiplication method** is used to obtain specific examples of elliptic curves $E$ over $\mathbb{F}_q$ with $E_q$ having a subgroup of order $m$.

# Some Families of Pairing-friendly Curves

- Some sparse families of ordinary pairing-friendly curves are:
  - **MNT (Miyaji-Nakabayashi-Takano) curves:** These are curves of prime orders with embedding degrees 3, 4 or 6.
  - **Freeman curves:** These curves have embedding degree 10.
- Some complete families of ordinary pairing-friendly curves are:
  - **BN (Barreto-Naehrig) curves:** These curves have embedding degree 12 and discriminant 3.
  - **SB (Scott-Barreto) curves**
  - **BLS (Barreto-Lynn-Scott) curves**
  - **BW (Brezing-Weng) curves**

# Efficient Implementation

- **Denominator elimination:** Let $k$ be even. Take $d = k/2$.
- $f_{n,P}(Q)$ is computed by Miller's algorithm, where $Q = (h, k)$ with $h \in \mathbb{F}_{q^d}$.
- The denominators $L_{2U,-2U}(Q)$ and $L_{U+P,-(U+P)}(Q)$ correspond to vertical lines, evaluate to elements of $\mathbb{F}_{q^d}$, and can be discarded.
- The final exponentiation guarantees correct computation of $\hat{e}_m(P, Q)$.

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- **BMX (Blake-Murty-Xu) refinements** use 2-bit windows in Miller's loop.

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- **Loop reduction:** With clever modifications to Tate pairing, the number of iterations in the Miller loop can be substantially reduced.
- A typical reduction is by a factor of 2.
- **Examples**
  - $\eta$ **and** $\eta_T$ **pairings**
  - **Ate pairing**
  - **R-ate pairing**

# References for Part II


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# Part III

## Hyperelliptic Curves

- Representation of the Jacobian

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