Exercise set 2 – Solutions

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1. Let H and H' be the two components of $T - e$ and let $F \subseteq E(T')$ consist of the edges of T' with one endpoint in $V(H)$, the other in $V(H')$. Since T' is connected, $F \neq \emptyset$. Furthermore, since T has the unique edge e joining H and H', $F \subseteq E(T') \setminus E(T)$. $T' + e$ contains a unique cycle C of which e is an edge. C leaves H and enters H' via e. In order to complete the cycle, one must use one edge e' of $E(T')$ to come back from H' to H. But then $e' \in F$. It is now clear that for this e' both $T - e + e'$ and $T' + e - e'$ are spanning trees of G. (Note that the cycle C, after coming back to H, may again enter H' and subsequently return back to H. Every time it does so, it has to use two new edges from F. That is, the choice of e' is not always unique.)

2. **(a)** [if] e is not a cut-edge of G. Then $G \setminus e$ is connected and hence has a spanning tree T. But then T is a spanning tree of G too and $e \notin E(T)$.

[only if] Let e be a cut-edge of G with endpoints u and v. The only u, v-path in G is the edge e (since another u, v-path in G produces a cycle in conjunction with e). Let T be a spanning tree in G. Since T is connected, T contains a u, v-path which has to be the edge e. Thus $e \in E(T)$.

(b) [if] Let e be a non-loop edge of G and let u be an endpoint of G. We can grow a BFS (or DFS) spanning tree of G rooted at u and containing the edge e.

[only if] A tree is a simple graph and hence does not contain a loop.

3. Let S be the set of all spanning trees of K_n. By Cayley's formula $|S| = n^{n-2}$. In order to get $\tau(K_n \setminus e)$ for a given $e \in E(K_n)$ we have to subtract from n^{n-2} the number k of spanning trees of K_n containing the particular edge e. Because of symmetry k is independent of the choice of e. Look at the sum $\sigma := \sum_{T \in S} e(T)$. Since each tree T in the sum has $n - 1$ edges, we have $\sigma = (n-1)n^{n-2}$. On the other hand, each edge of K_n is counted k times in the above sum, so that $\sigma = k \times e(K_n) = kn(n-1)/2$. Equating the two expressions for σ gives $k = 2n^{n-3}$. Thus $\tau(K_n \setminus e) = n^{n-2} - k = (n-2)n^{n-3}$.

4. Let the partite sets of $K_{s,t}$ be X and Y with $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$. The Q-matrix under the vertex ordering $x_1, \ldots, x_s, y_1, \ldots, y_t$ is then

$$Q = \begin{pmatrix} t & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & t & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & s & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & s \end{pmatrix}.$$

Let us choose to delete the first row and the first column of Q to get

$$\tau(K_{s,t}) = \begin{vmatrix} t & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & t & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & s & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & s \end{vmatrix}.$$

$$\tau(K_{s,t}) = \begin{vmatrix} t & 0 & \cdots & 0 & -t & 0 & 0 & \cdots & 0 \\ 0 & t & \cdots & 0 & -t & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t & -t & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & t & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & -1 & s & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & -1 & 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & -1 & 0 & 0 & \cdots & s \end{vmatrix}.$$

Adding $1/t$ times each of the first $s-2$ rows to each of the last t rows gives:

$$\tau(K_{s,t}) = \begin{vmatrix} t & 0 & \cdots & 0 & -t & 0 & 0 & \cdots & 0 \\ 0 & t & \cdots & 0 & -t & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t & -t & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & t & -1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & -(s-1) & s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1) & 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -(s-1) & 0 & 0 & \cdots & s \end{vmatrix}.$$

Adding $1/s$ times each of the last t rows to the $(s-1)$-st row then yields

$$\tau(K_{s,t}) = \begin{vmatrix} t & 0 & \cdots & 0 & -t & 0 & 0 & \cdots & 0 \\ 0 & t & \cdots & 0 & -t & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t & -t & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & t/s & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1) & s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1) & 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -(s-1) & 0 & 0 & \cdots & s \end{vmatrix}.$$

Finally we add s times the $(s-1)$-st row to each of the first $s-2$ rows to get

$$\tau(K_{s,t}) = \begin{vmatrix} t & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & t & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & t/s & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1) & s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1) & 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -(s-1) & 0 & 0 & \cdots & s \end{vmatrix}.$$

Now $\tau(K_{s,t})$ is the determinant of a lower triangular matrix and hence is equal to the product of the entries in the main diagonal of this matrix, i.e., $\tau(K_{s,t}) = t^{s-2}(t/s)s^t = s^{t-1}t^{s-1}$.

5. We will prove that G is the cycle C_n. Since $G \setminus u$ is a tree for every $u \in V(G)$ and $n(G) \geqslant 3$, G does not contain multiple edges and loops, i.e., G is simple. If G has more than two components, then deleting a vertex u in one component leaves other components unaffected, i.e., leaves the graph disconnected, a least 2 vertices. Again deleting a vertex $u \in V(H)$ leaves a disconnected graph that cannot be a tree. Thus G has only one component, i.e., G is connected.

By hypothesis G has no cut vertices and hence no cut edges (Exercise 14), i.e., every edge of G lies on a cycle. If G contains more than one cycle, we can choose two cycles C_1 and C_2 of G and a vertex $u \in V(C_1) \setminus V(C_2)$. But then $G \setminus u$ contains the cycle C_2 and hence is not a tree. Therefore, G contains exactly one cycle.

6. **(a)** If any two of u, v, w lie in different components of G, then the triangle inequality clearly holds. So assume that all of these three points lie in the same component of G. Let P be a shortest u, v-path and Q a shortest v, w-path. The u, w-walk consisting of P followed by Q contains a u, w-path R. Hence $d(u, w) \leqslant |R| \leqslant |P_1| + |P_2| = d(u, v) + d(v, w)$.

(b) Let $\operatorname{rad} G = \epsilon(u)$ and $\operatorname{diam} G = \epsilon(u') = d(u', v')$. But then by the triangle inequality $\operatorname{diam} G \leqslant d(u', u) + d(u, v') \leqslant \epsilon(u) + \epsilon(u) = 2\epsilon(u) = 2 \operatorname{rad} G$.

(c) If $r = d = 1$, consider P_2. If $r = 1$ and $d = 2$, consider P_3. So assume that $r > 1$. Take G to be the connected graph that decomposes into a cycle $C \cong C_{2r}$ and a path $P \cong P_{d-r+1}$ such that C and P share exactly one vertex u which is also an endpoint of P. Let v be the other endpoint of P. (If $d = r$, then $v = u$.) One can readily check that $\operatorname{rad} G = \epsilon(u) = r$ and $\operatorname{diam} G = \epsilon(v) = (d - r) + r = d$.

7. **(a)** $G = K_n$. (Any two vertices of G do not form an independent set.)

(b) The only non-trivial component of G is either a star or a 3-cycle. (Let $M := \{uv\}$ be a maximal matching of G. Let n_1 (resp. n_2) be the number of neighbors of u (resp. v) other than v (resp. u), which are not neighbors of v (resp. u). Also let n_3 be the number of common neighbors of u and v. One can easily check that if at least two of n_1, n_2, n_3 are non-zero, then M is not maximal. Also if $n_1 = n_2 = 0$, we must have $n_3 = 1$.)

(c) The only non-trivial component of G is a star. (Every edge of G is incident to the vertex in a minimum vertex cover.)

(d) G is an edge. (Every vertex of G is incident to the edge in a minimum edge cover.)

8. **(a)** Let G be an X, Y-bigraph.

[if] By hypothesis $|N(S)| \geqslant |S|$ for any $S \subseteq X$; so G has a matching that saturates X, i.e., $|Y| \geqslant |X|$. By reversing the roles of X and Y one can similarly prove that $|X| \geqslant |Y|$.

[only if] A perfect matching of G saturates both X and Y and hence by Hall's theorem $|N(U)| \geqslant |U|$ for every $U \subseteq X$ and also $|N(V)| \geqslant |V|$ for every $V \subseteq Y$. Given $S \subseteq V(G)$ one can write S as the disjoint union $U \cup V$, where $U = S \cap X$ and $V = S \cap Y$. The neighborhoods of U and V are also disjoint and so $|N(S)| = |N(U)| + |N(V)| \geqslant |U| + |V| = |S|$.

(b) The complete graph K_{2n+1}, $n \in \mathbb{N}$, satisfies

$$|N(S)| = \begin{cases} 0 & \text{if } S = \emptyset, \\ 2n & \text{if } |S| = 1, \\ 2n+1 & \text{if } |S| \geqslant 2, \end{cases}$$

that is, $|N(S)| \geqslant |S|$ for any $S \subseteq V(K_{2n+1})$. Since K_{2n+1} contains an odd number of vertices, it cannot have a perfect matching.

9. Let S be a maximum independent set in G (so that $|S| = \alpha(G)$) and let $T := V(G) \setminus S$. Since S is an independent set, the sum $\sum_{v \in T} d(v)$ counts each edge of G at least once, i.e., $e(G) \leqslant \sum_{v \in T} d(v) \leqslant (n(G) - \alpha(G))\Delta(G)$. Rearranging gives $\alpha(G) \leqslant n(G) - e(G)/\Delta(G)$.

If G is regular, by the degree sum formula we get $2e(G) = n(G)\Delta(G)$, i.e., $e(G)/\Delta(G) = n(G)/2$, so that $\alpha(G) \leqslant n(G) - e(G)/\Delta(G) = n(G)/2$.

in M (i.e. the set of vertices saturated by M). Q covers all the edges of G, for, if not, then G has an edge with unsaturated endpoints u and v. But then adding this edge to M will give a matching bigger than M, a contradiction.

For every $k \in \mathbb{N}$ we have $\alpha'(kP_3) = k$ and $\beta(kP_3) = 2k$.

11. The statement is true. The proof follows from the fact that a tree is acyclic and from the following lemma.

Lemma: Let G be a graph with two distinct perfect matchings M and M'. Then every component of the symmetric difference $M \triangle M'$ is an (even) cycle.

Proof Every component of $M \triangle M'$ is either an even cycle or a path, the edges in which alternate between $M \setminus M'$ and $M' \setminus M$. Let u be an endpoint of such a path P. In view of symmetry we can assume that the edge e of P incident on u belongs to $M \setminus M'$. The only edge incident to a vertex of degree 1 must belong to every perfect matching. Thus $d_T(u) \geqslant 2$. Since M' saturates u anyway, we can choose a neighbor v of u not in P such that $uv \in M'$. Since $e \in M$, $uv \notin M$. Thus $uv \in M' \setminus M$. This implies that P cannot be a component of $M \triangle M'$. •

12. [if] Define a function $f : V(T) \to V(T)$ as follows. Choose $v \in V(T)$. By hypothesis $T \setminus v$ has only one odd component, call it H. If v has two distinct neighbors u_1, u_2 in H, then a u_1, u_2-path in H forms a cycle with the edges vu_1 and vu_2. Thus there exists a unique neighbor u of v in H. Define $f(v) := u$.

First we claim that f is injective. Assume not, i.e., $f(v_1) = f(v_2) = u$ for some $v_1 \neq v_2$. Think of T as a tree rooted at u. Let the children of u be $v_1, v_2, \ldots, v_k$. Call T_i the subtree of T rooted at v_i. Since $f(v_1) = u$, every subtree rooted at a child of v_1 is of even order. Therefore, $n(T_1)$ is odd. Similarly, $n(T_2)$ is also odd. But then $T \setminus u$ has at least two odd components (T_1 and T_2), a contradiction to the hypothesis.

Next we claim that f^2 is the identity map on $V(T)$. Choose $v \in V(T)$. Since $V(T)$ is finite, the elements $v, f(v), f^2(v), f^3(v), \ldots$ cannot be all distinct. Choose $0 \leqslant k < l$ such that $f^k(v), \ldots, f^{l-1}(v)$ are pairwise distinct, but $f^k(v) = f^l(v)$ for some $l > k$. Since T does not contain loops, $l > k+1$. If $l > k+2$, then $(f^k(v), f^{k+1}(v), \ldots, f^{l-1}(v))$ is a cycle in T, a contradiction. So $l = k+2$, i.e., $f^{k+2}(v) = f^k(v)$. Since f is a map from a finite set to itself, its injectivity implies its bijectivity. Applying the function f^{-k} in the last equation gives $f^2(v) = v$, as claimed.

Thus f produces the desired pairing of vertices for a perfect matching.

[only if] Let T have a perfect matching. By Tutte's 1-factor theorem $o(T \setminus v) \leqslant 1$. If $o(T \setminus v) = 0$, then T contains an odd number of vertices. No graph with an odd number of vertices can have a perfect matching.

13. Choose (nonempty) $S \subseteq V(G)$ and count the number k of edges from S to the odd components of $G \setminus S$ (as in the case of a corollary proved in the class). G being 3-regular, counting such edges using their endpoints in S gives $k \leqslant 3|S|$.

Let $H_1, \ldots, H_l$ be all the odd components of $G \setminus S$ (where $l = o(G \setminus S)$) and let l_i be the number of edges between H_i and S. The degree sum formula for H_i and the 3-regularity of G yield $2e(H_i) = 3n(H_i) - m_i$. Since $n(H_i)$ is odd, we then have m_i. Finally since G has at most two cut-edges $m_i = 1$ for at most two i and $m_i \geqslant 3$ otherwise. Therefore, $k \geqslant 3o(G \setminus S) - 4$.

Combining the two inequalities involving k yields $o(G \setminus S) \leqslant |S| + 4/3$. Since $o(G \setminus S)$ and $|S|$ are integers, the last inequality implies $o(G \setminus S) \leqslant |S| + 1$. Assume that $o(G \setminus S) = |S| + 1$. It is a straightforward check that in this case $n(G)$ is odd. But the degree sum formula for G gives $2e(G) = 3n(G)$, i.e., $n(G)$ is even, a contradiction. Thus $o(G \setminus S) \leqslant |S|$. Now apply Tutte's theorem.

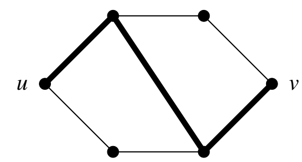
14. P_2 (an edge) provides a counterexample to the given statement. The corrected assertion is: *Let e be a cut-edge in G. If the component of G containing e has more than two vertices, then at least one endpoint of G is a cut vertex.*

For the proof of the corrected assertion, let H be the component of G containing e and let u and v be the endpoints of e. Further let H_1 and H_2 be the two components of $H \setminus e$ with $u \in V(H_1)$ and $v \in V(H_2)$. Since $n(H) \geqslant 3$, (at least) one of H_1 and H_2 has (at least) two vertices. Because of symmetry we can separates w from v.

15. Choose two non-adjacent vertices in the Petersen graph G. These two vertices have the label ab and ac for some permutation a, b, c, d, e of $1, 2, 3, 4, 5$. G contains three pairwise internally disjoint ab, ac-paths: i) ab, de, ac, ii) ab, cd, be, ac, and iii) ab, ce, bd, ac. By Menger's theorem G is 3-connected, i.e, $\kappa(G) \geqslant 3$. Also $\kappa(G) \leqslant \delta(G) = 3$. Thus $\kappa(G) = 3$. But then $\kappa'(G) = 3$ (since $\kappa(G) \leqslant \kappa'(G) \leqslant \delta(G)$, or since a 3-regular graph H has $\kappa(H) = \kappa'(H)$).

16. Let $F = [S, \overline{S}]$ and $F' := [S', \overline{S'}]$ be two different edge cuts of G. Then $S \triangle S'$ is a non-empty proper subset of $V(G)$. One can readily verify that $F \triangle F' = [T, \overline{T}]$.

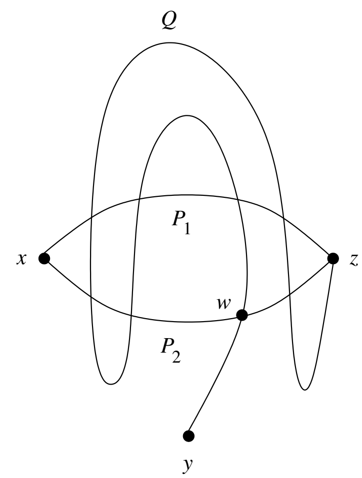
17. The statement is false. A counterexample is provided by the 2-connected graph shown in the adjacent figure. Consider the u, v-path P shown by bold edges. The graph contains no u, v-path internally disjoint from P.



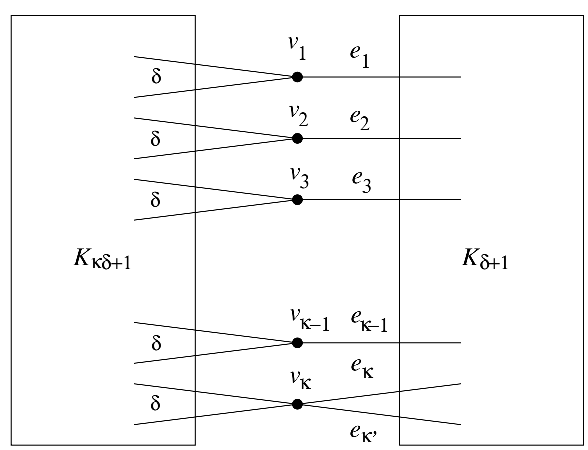
18. [if] Clearly G is connected (By hypothesis G satisfies a condition stronger than mere connectivity). Choose $x \in V(G)$. We will show that $G \setminus x$ is connected. Choose $y, z \in V(G \setminus x)$. By hypothesis there exists an x, y-path P in G through z. Deletion of x retains the part of P from z to y.

[only if] Suppose that G is 2-connected and choose three pairwise distinct vertices $x, y, z \in V(G)$. By Menger's theorem G contains two internally disjoint x, z-paths P_1 and P_2. Furthermore, $G \setminus x$ contains a y, z-path Q. Let w be the vertex on Q, that is nearest to y and that is also in $V(P_1) \cup V(P_2)$. By symmetry let us assume that $w \in V(P_2)$. (Note that we can have $w = z$, in which case w belongs to $V(P_1)$ as well. Also $w \neq x$.)

P_1, the z, w-subpath of P_2 and the w, y-subpath of Q give us an x, y-path through z.



19. Consider the graph G shown in the adjacent figure. The vertices $v_1, \ldots, v_\kappa$ each has pairwise distinct sets of δ neighbors in $K_{\kappa\delta+1}$. $v_1, \ldots, v_{\kappa-1}$ has one neighbor each in $K_{\delta+1}$, whereas v_κ has $\kappa' - \kappa + 1$ neighbors in $K_{\delta+1}$. All these neighbors in $K_{\delta+1}$ are pairwise distinct.



One can now verify that $\delta(G) = \delta$ (look at the vertex in $K_{\delta+1}$ having no neighbor in $\{v_1, \ldots, v_\kappa\}$).

Consider a pair (x, y) of non-adjacent vertices in this graph. If $x = v_i$ and $y = v_j$ for some $i \neq j$, then there exist $\delta \geqslant \kappa$ internally disjoint v_i, v_j-paths via $K_{\kappa\delta+1}$. If $x = u \in V(K_{\kappa\delta+1})$ and $y = v_i$, we again get $\delta \geqslant \kappa$ internally disjoint u, v_i-paths via $K_{\kappa\delta+1}$. If $x = v_i$ and $u = w \in V(K_{\delta+1}$, we get κ internally disjoint v_i, w-paths, one via e_i, the others via $K_{\kappa\delta+1}, v_j$ and e_j for each $j \neq i$. Finally, if $x = u \in V(K_{\kappa\delta+1})$ and $y = w \in V(K_{\delta+1})$, then every u, w-path must consist of one of the vertices $v_1, \ldots, v_\kappa$. Also we can arrange exactly κ u, w-paths one through each v_i. By Menger's theorem $\kappa(G) = \kappa$ and $\{v_1, \ldots, v_\kappa\}$ is a minimum vertex-cut of G.

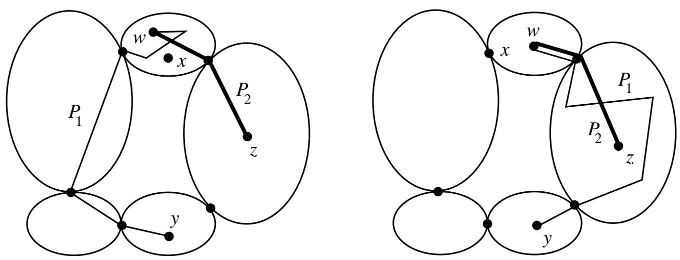
A similar study shows that $\kappa'(G) = \kappa'$ and $\{e_1, \ldots, e_{\kappa'}\} = [S, \overline{S}]$, $S = V(K_{\delta+1})$, is a minimum edge-cut of G. (Use the Elias-Feinstein-Shannon-Ford-Fulkerson theorem.)

vertices of G. Let us represent the corresponding vertices in $B(G)$ by $b_1, \ldots, b_s$ and $u_1, \ldots, u_t$ respectively. If $t = 0$, then $B(G)$ is the single-vertex tree and has no leaf. So let us concentrate on the case that $t > 0$, so that $s > 1$.

Let B be a block of G. Since G is connected and B does not contain all the vertices of G, there exists $x \in V(B)$ such that x has a neighbor y outside B. The edge xy lies in a separate block B' of G. But then B and B' share x and so x is a cut vertex of G. Thus every block contains at least one cut vertex.

By definition $B(G)$ is simple. In order to prove that $B(G)$ is connected, it suffices to produce a b_i, u_j-path for every i, j. B_i contains a cut vertex of G, say, v_k. Since G is connected, there is a v_k, v_j-path P in G. Let us follow the path P starting from v_k. We simultaneously generate a b_i, u_j-walk Q. Initially we are at v_k in P and b_i in Q. Assume that at some stage we are at x in P and b_l in Q. If x is not a cut-vertex of G, we remain at b_l in Q and if P is not yet finished, we proceed to the next vertex on P. If x is a cut-vertex, say v_m, we move from b_l to u_m in Q, and if P is not finished and the next edge on P belongs to B_n, we move from u_m to b_n and proceed to the next vertex on P. The resulting b_i, u_j-walk Q contains a b_i, u_j-path.

Now suppose that $(b_{i_1}, u_{i_1}, \ldots, b_{i_r}, u_{i_r})$ is a cycle in $B(G)$. Since $B(G)$ is simple, $r \geqslant 3$. I will show that $B := \bigcup_{k=1}^{r} B_{i_k}$ is 2-connected, contradicting the maximality of each block B_{i_k}. Choose $x \in V(B)$ and $y, z \in V(B) \setminus \{x\}$. By symmetry we can assume that $x \in V(B_{i_1})$. Since B_{i_1} is a block and has more than one vertex (G contains no isolated vertex), $B_{i_1} \setminus x$ is connected. Choose $w \in V(B_{i_1}) \setminus x$. Then there is a y, w-path P_1 and a w, z-path P_2 in $B \setminus x$. P_1 and P_2 produce a y, z-walk which contains a y, z-path. (See the following figure explaining two possibilities for x. P_1 (resp. P_2) may lie entirely in B_{i_1}, if y (resp. z) is in $V(B_{i_1}) \setminus \{x\}$. If both y and z are in $V(B_{i_1}) \setminus \{x\}$, then one directly gets an y, z-path in the connected graph $B_{i_1} \setminus x$.)



Thus $B(G)$ is a tree. What remains is to show that a cut vertex of G must belong to at least two blocks of G. Assume the contrary, that is, a cut vertex x belongs to only one block B of G. If x has a neighbor y outside B, then the edge xy belongs to a different block B' so that x belongs to B' as well. So x does not have a neighbor outside B. Now choose any $y, z \in V(G \setminus x)$. Since G is connected, there is a y, z-path P in G. If P does not involve B, it remains in $G \setminus \{x\}$. Otherwise let y' and z' be the first and last vertices of P which are in $V(B)$. (We might have $y' = y$ and/or $z' = z$.) Since B is 2-connected, there is a y', z'-path Q in $B \setminus x$. The y, y'-subpath of P, Q and the z', z-subpath of P give a y, z-path in $G \setminus \{x\}$. Thus $G \setminus \{x\}$ is connected, i.e., x is not a cut vertex of G, a contradiction.

21. We start by proving that for any plane graph H the dual H^* is connected. For the proof let X_u be the unbounded face of H and X any other face of H. If we draw a (semi-infinite) ray from the point $x \in V(H^*)$ representing X, the ray will eventually go the interior of X_u (because all faces of H other than X_u are bounded). From a point on the ray in the interior of X_u there is a polygonal curve ending in x_u (the vertex representing X_u) and lying entirely inside X_u. We can distort the ray, if necessary, to get a simple polygonal curve C starting from x and ending in x_u such that C does not go through any vertex of H. C crosses a finite number of edges of H (at their internal points). Every time it does so, it may or may not move from one face of H to another. In any case, C traverses a finite sequence of faces of H, in which two consecutive faces share an edge. This gives us an x, x_u-path in H^*. But then for any $x, x' \in V(H^*)$ an x, x_u- and an x', x_u-path yield an x, x'-walk and hence an x, x'-path in H^*. Thus H^* is connected, as claimed.

First assume that G is disconnected. Taking $H = G^*$ in the last paragraph shows that $H^* = (G^*)^*$ is connected and hence cannot be the same as G.

For the converse, let G be connected. Let us draw G^* in such a way that each edge of G^* crosses only the corresponding edge of G (only once) and no other edge of G or G^*. Now let X^* be a face of G^*. Any edge the face X^* and must terminate before leaving X^*. That is, X^* contains at least one vertex of G. Assume that some face of G^* contains more than one vertex of G, i.e., $n > f^*$. But $f = n^*$ and $e = e^*$. Since G^* is connected, Euler's formula gives $n^* - e^* + f^* = 2$. Combining all these findings gives $n - e + f > 2$, i.e., G is not connected, a contradiction. Thus every face of G^* contains exactly one vertex of G. But then we can use each e as $(e^*)^*$ and obtain G as the dual of $(G^*)^*$.

22. It is sufficient to prove the assertion for a connected graph, since otherwise we could add an appropriate number of edges and prove the assertion for the resulting connected outerplane graph. The length of the outer face of G is n, whereas the length of every other face is at least 3 (since G is simple). Therefore, the degree sum formula for G^* gives $2e = 2e^* \geqslant n + 3(f-1)$. Since G is plane also, $n - e + f = 2$. Eliminating f gives $e \leqslant 2n - 3$.

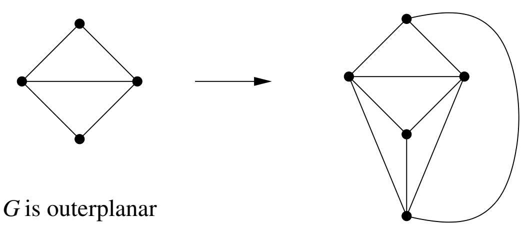
23. Consider the following two cases:

Case 1: G is not planar.

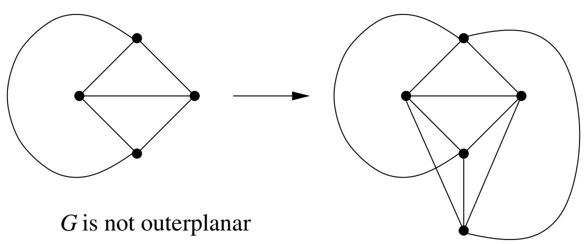
Then G is clearly non-outerplanar too. Also G contains a subdivision of K_5 or $K_{3,3}$ and hence a subdivision of K_4 or $K_{2,3}$.

Case 2: G is planar.

Consider the graph H obtained by adding a new vertex to G and joining this new vertex to every vertex in G. It follows that G is outerplanar $\iff$ H is planar $\iff$ H contains no subdivision of K_5 or $K_{3,3}$ $\iff$ G contains no subdivision of K_4 or $K_{2,3}$. This is exemplified in the following figure:



G is outerplanar



G is not outerplanar

24. **(a)** It is sufficient to prove the assertion for connected graphs. By Euler's formula $n - e + f = 2$. Since G has girth k, every face of G has length at least k, i.e., $2e \geqslant kf$. Eliminating f gives $e \leqslant (n-2)\frac{k}{k-2}$.

(b) The Petersen graph has girth 5. By Part (a) every simple planar graph with 10 vertices and of girth 5 contains at most $8 \times 5/3 = 13.333\ldots$, i.e., at most 13 edges. But the Petersen graph has 15 edges.

25. **(a)** Let $G \cong G^*$. Since G^* is connected (Exercise 21), G is also connected. By Euler's formula $n - e + f = 2$. But the isomorphism $G \cong G^*$ implies $n = n^* = f$, i.e., $n - e + n - 2$, i.e., $e = 2n - 2$.

(b) Let G be the cycle C_{n-1} plus another vertex joined to each vertex of C_{n-1}. Then $G \cong G^*$. The adjacent figure explains the construction for $n = 7$.