1. Prove or disprove: The complement of a simple disconnected graph must be connected.

**Solution** The statement is true. Let $G$ be a simple disconnected graph and $u, v \in V(G)$. If $u$ and $v$ belong to different components of $G$, then the edge $uv \in E(G)$. If $u$ and $v$ belong to the same component of $G$, choose a vertex $w$ in another component of $G$. ($G$ has at least two components, since it is disconnected.) But then the edges $uw$ and $wv$ belong to $E(G)$. That is, in all cases there is a $u, v$-path in $G$.  

2. Prove that a bipartite graph has a unique bipartition (apart from interchanging the partite sets) if and only if it is connected.

**Solution** [if] Let $G$ be a bipartite graph and choose $v \in V(G)$. Let $X, Y$ and $X', Y'$ be two different bipartitions of $G$ with $v \in X$ and $v \in X'$. Since $X \neq X'$, at least one of $X \setminus X'$ and $X' \setminus X$ is non-empty. Also $Y \neq X'$, so both $Y \setminus X'$ and $X' \setminus Y$ can not be empty. Finally $X, X', Y$ and $Y'$ are independent sets. It follows that $G$ is the disjoint union of two bipartite graphs, one with bipartition $X \setminus X', X' \setminus X$ and the other with bipartition $Y \setminus X', X' \setminus Y$. So $G$ is not connected.

[only if] Let $G$ be a disconnected bipartite graph, $H$ a component of $G$ and $H'$ the disjoint union of all other components of $G$. $H$ and $H'$ are again bipartite, say, with bipartitions $X, Y$ and $X', Y'$ respectively. Then $X \cup X', Y \cup Y'$ and $X \cup Y', Y \cup X'$ are two different bipartitions of $G$.  

3. Prove or disprove: Every Eulerian bipartite graph contains an even number of edges.

**Solution** The statement is true. For the proof let $G$ be an Eulerian bipartite graph with bipartition $X, Y$ of its non-trivial component. We can count the number of edges in $G$ as $e(G) = \sum_{v \in X} d(v)$. Since $G$ is Eulerian, each degree $d(v)$ in this sum is even.  

4. Which of the following is a graphic sequence: $(5, 5, 5, 4, 2, 1, 1, 1)$ and $(5, 5, 4, 4, 2, 1, 1, 1)$? If it is graphic, produce a realization of the sequence, else prove why it is not graphic.

**Solution** $(5, 5, 5, 4, 2, 1, 1, 1)$ is graphic $\iff$ $(4, 4, 3, 1, 0, 1, 1)$, i.e., $(4, 4, 3, 1, 1, 1, 0)$ is graphic $\iff$ $(3, 2, 0, 0, 1, 0)$, i.e., $(3, 2, 1, 0, 0, 0)$ is graphic. The last sequence is not graphic, since a vertex of degree 3 in a simple graph must have three neighbors each of positive degree.

$(5, 5, 4, 4, 2, 1, 1, 1)$ is graphic $\iff$ $(4, 3, 3, 1, 1, 1, 1)$ is graphic $\iff$ $(2, 2, 0, 0, 1, 1)$, i.e., $(2, 2, 1, 1, 0, 0)$ is graphic $\iff$ $(1, 0, 1, 0, 0)$, i.e., $(1, 1, 0, 0, 0)$ is graphic $\iff$ $(0, 0, 0, 0)$ is graphic. But four isolated vertices realize $(0, 0, 0, 0)$ and so $(5, 5, 4, 4, 2, 1, 1, 1)$ is graphic. Starting from four isolated vertices one can work back and come up with the realization shown in the adjacent figure.  

5. Let $n$ be a positive integer of the form $4k$ or $4k + 1$ for some $k \in \mathbb{N}$. Construct a simple graph $G$ with $n$ vertices, $n(n - 1)/4$ edges and with $\Delta(G) - \delta(G) \leq 1$.

**Solution** $[n = 4k]$ Let $H$ be the complete bipartite graph with bipartition $\{u_1, \ldots, u_{2k}\}, \{v_1, \ldots, v_{2k}\}$. Take $G := H \setminus \{u_iv_i \mid i = 1, \ldots, k\}$.

$[n = 4k + 1]$ Start with the graph $G$ constructed for the case $n = 4k$. Add to $G$ a new vertex $w$ and the edges $wu_i$ and $wv_i$ for each $i = 1, \ldots, k$. The resulting graph is $2k$-regular.  

6. Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers.

**Solution** Consider the acquaintance graph $G$ of a given set of six people with vertex set $\{v_1, \ldots, v_6\}$. We have to show that either $G$ or $\overline{G}$ has a triangle. Since $K_6$ decomposes into $G$ and $\overline{G}$, $v_1$ has a total of five
Let \( v \) be a cut-vertex of a simple graph \( G \). Prove that \( \overline{G} \setminus v \) is connected.

**Solution** First note that \( \overline{G} \setminus v \) is the same as \( G \setminus v \). Now use Exercise 1.

11. Let \( G_n \) be the simple graph with each vertex labeled by a permutation of \( \{1, 2, \ldots, n\} \) and with two vertices adjacent if and only if their labels differ by an interchange of two adjacent entries. Prove that \( G_n \) is connected.

**Solution** Let \( (i_1, i_2, \ldots, i_n) \) and \( (j_1, j_2, \ldots, j_n) \) be two permutations of \( \{1, 2, \ldots, n\} \). By a sequence of interchanges of adjacent entries one can convert \( (i_1, i_2, \ldots, i_n) \) to \( (1, 2, \ldots, n) \) — as in the bubble-sort algorithm. (This fact can be easily proved by induction on \( n \).) Similarly a sequence of interchanges of adjacent entries produces \( (1, 2, \ldots, n) \) from \( (j_1, j_2, \ldots, j_n) \). If one reverses this second sequence of interchanges, one gets back \( (j_1, j_2, \ldots, j_n) \) from \( (1, 2, \ldots, n) \).

12. Let \( G \) be a connected simple graph not containing \( P_4 \) or \( C_3 \) as an induced subgraph. Prove that \( G \) is a biclique (i.e., a complete bipartite graph).

**Solution** We first prove by induction on \( k \in \mathbb{N} \) that \( G \) contains no cycles of length \( 2k + 1 \). Since \( G \) does not contain \( C_3 \) as (induced) subgraph, \( G \) does not contain 3-cycles. So consider \( k \geq 2 \) and suppose that \( G \) does not contain cycles of length \( 3, 5, \ldots, 2k - 1 \). Assume that \( G \) contains a \( 2k + 1 \)-cycle \( Z \). If \( G \) contains an edge \( e \) between non-adjacent vertices of \( Z \), then \( G \) contains two smaller cycles \( Z_1 \) and \( Z_2 \). We choose \( e \) in such a way that the length of \( Z_1 \) is minimum. Now by the induction hypothesis neither \( Z_1 \) nor \( Z_2 \) is odd,
13. Let $P$ and $Q$ be paths of maximum length in a connected graph $G$. Prove that $V(P) \cap V(Q) = \emptyset$.

\textit{Solution} Assume that $V(P) \cap V(Q) = \emptyset$. Let the endpoints of $P$ be $u$ and $v$ and those of $Q$ be $x$ and $y$. Since $G$ is connected, there is a $u, x$-path $R$ in $G$. Since $P$ and $Q$ do not share a vertex, there is a portion $S$ of $R$ such that $S$ is a $p, q$-path with $p \in V(P)$ and $q \in V(Q)$ and such that $S$ does not contain any other vertex of $P$ or $Q$. Let $l := |P| = |Q|$. Either the $u, p$- or the $p, v$-subpath of $P$ is of length $\geq l/2$. Because of symmetry we may assume that the former path, call it $P'$, is not shorter than the latter. Similarly assume that the $x, q$-subpath $Q'$ of $Q$ is not shorter than the $q, y$-subpath of $Q$. That is, $|P'| \geq l/2$ and $|Q'| \geq l/2$. Now $P'$ followed by $S$ followed by $Q'$ is a $u, x$-path of length $\geq (l/2) + 1 + (l/2) > l$, a contradiction to the maximality of $|P| = |Q|$.

14. Prove that an even graph has no cut edge. For each $k \in \mathbb{N}$ produce a $2k + 1$-regular simple graph with a cut edge.

\textit{Solution} Let $G$ be an even graph and let $e \in E(G)$ have endpoints $u$ and $v$. Deleting $e$ from $G$ makes the degrees of $u$ and $v$ odd and leaves the degrees of other vertices unchanged (namely even). If $u$ and $v$ belong to different components in $G \setminus e$, then the component of $G \setminus e$ containing $u$ would have exactly one odd vertex, a contradiction. So $u$ and $v$ are connected in $G \setminus e$. A $u, v$-path in $G \setminus e$ forms a cycle in conjunction with $e$, i.e., $e$ lies on a cycle in $G$. Thus $e$ is not a cut-edge.

The adjacent graph is $2k + 1$-regular and simple with cut edge $wx$.

15. Deduce that the total number of simple even graphs on a given set of $n$ vertices is $2^{\binom{n-1}{2}}$.

\textit{Solution} Let $S$ be the set of all even simple graphs on the vertex set $\{v_1, \ldots, v_n\}$. Also let $T$ be the set of all simple graphs on the vertex set $\{v_1, \ldots, v_{n-1}\}$. I will establish a bijection between $S$ and $T$. Consider the map $f : S \rightarrow T$ defined by $G \mapsto G \setminus v_n$. For the converse consider $g : T \rightarrow S$ defined as follows. Take a graph $H \in T$ and let $U$ denote the set of all odd vertices in $H$. Add to $H$ the vertex $v_n$ and an edge $v_i v_n$ for every $v_i \in U$. Since $|U|$ is even, the resulting graph $(\in S)$ is even and is defined to be $g(H)$. It’s easy to argue that $g \circ f = \text{id}_S$ and $f \circ g = \text{id}_T$, i.e., $f$ is a bijection.

16. Argue that the Petersen graph has exactly 12 five-cycles.

\textit{Solution} Call the Petersen graph $G$. First I claim that every edge $l$ in $G$ belongs to exactly four five-cycles. For the proof let the endpoints of $l$ be $ab$ and $cd$ for some $\{a, b, c, d\} \subset \{1, 2, 3, 4, 5\}$. (Since $ab$ and
Let \( n \) be the number of 5-cycles in \( G \). Summing up the edges in all these cycles gives a total of \( 5n \) edges. In this sum each edge of \( G \) is counted four times in view of the previous claim. Since \( G \) has 15 edges, we have \( 5n = 4 \times 15 = 60 \), i.e., \( n = 12 \). 

17. Let \( G \) be an \( n \)-vertex simple graph with \( n \geq 2 \). Determine the maximum possible number of edges in \( G \) under each of the following conditions:

i) \( G \) has an independent set of size \( k \).

ii) \( G \) has exactly \( k \) components.

iii) \( G \) is disconnected.

**Solution**

i) \( \binom{n}{2} - \binom{k}{2} \). Achieved by the graph obtained by deleting from \( K_n \) all edges with both endpoints in a chosen \( k \)-element subset of \( V(K_n) \).

ii) \( (\binom{n-k+1}{2}) \). Let \( H_1, \ldots, H_k \) be the \( k \) components of \( G \) with \( n_i := n(H_i) \) for \( i = 1, \ldots, k \). We may assume (after rearranging, if necessary) that \( n_1 \geq n_2 \geq \cdots \geq n_k \geq 1 \). Then \( e(G) = e(H_1) + e(H_2) + \cdots + e(H_k) \leq \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2} = (n_1^2 + n_2^2 + \cdots + n_k^2 - n)/2 \). This number of edges corresponds to that of a disjoint union of \( k \) complete graphs \( (K_{n_1} + K_{n_2} + \cdots + K_{n_k}) \). Now for \( n_i \geq 1 \) we have \( e(K_{n_1+1} + K_{n_2+1} + \cdots + K_{n_k+1}) - e(K_{n_1} + K_{n_2} + \cdots + K_{n_k}) = (n_1 - n_i) + 1 \geq 1 \), i.e., we gain at least an edge by pushing a vertex from the \( i \)-th component to the first one. Thus if \( n_2 = \cdots = n_k = 1 \) (so that \( n_1 = n - k + 1 \)), we have the maximum value of \( e(G) \), i.e., \( e(G) \leq \binom{n-k+1}{2} \). This bound is achieved by \( K_{n-k+1} + K_1 + \cdots + K_1 \).

iii) \( \binom{n}{2} - \binom{k}{2} \). A disconnected graph \( G \) has \( k \geq 2 \) components and hence by Part ii) can have a maximum of \( \binom{n-k+1}{2} \) edges. This number is maximized for \( k = 2 \), which implies that \( n(G) \leq \binom{n}{2} \). This bound is achieved by \( K_{n-1} + K_1 \).

18. Show that every simple graph with at least two vertices contains at least two vertices of the same degree. Does the result continue to hold if `simple' is replaced by `loopless'?

**Solution** Let \( n \geq 2 \) and \( G \) a simple graph with \( n \) vertices. Since the vertex degrees can assume only \( n \) possible values \( 0, 1, \ldots, n - 1 \), the \( n \) vertices of \( G \) have pairwise distinct degrees if and only if \( G \) has a vertex \( v_i \) of degree \( i \) for each \( i = 0, \ldots, n - 1 \). But then \( v_{n-1} \) is adjacent to every other vertex in \( V(G) \) and in particular to \( v_0 \), a contradiction.

The result does not necessarily hold for loopless graphs. For example consider the adjacent 3-vertex loopless graph with the degree sequence \( (3, 2, 1) \).

19. Let \( n, s, t \) be non-negative integers with \( n = s + t > 0 \). Find necessary and sufficient conditions on \( n, s, t \) such that there exists a connected simple \( n \)-vertex graph with \( s \) vertices of odd degree and \( t \) vertices of even degree.

**Solution** The necessary and sufficient condition sought for is that \( s \) be even. The necessity is an immediate consequence of the degree-sum formula. For the converse let us be given an even \( s \) and any \( t \). If \( s = 0 \), then consider the cycle \( C_t \). If \( t = 0 \), consider the complete graph \( K_s \). Finally suppose that \( s > 0 \) and \( t > 0 \) and start with the complete biclique \( K_s,t \). If \( t \) is odd, \( K_s,t \) is a desired graph. Otherwise let \( X = \{u_1, \ldots, u_s\}, Y = \{v_1, \ldots, v_t\} \) be the bipartition of the vertices of \( K_s,t \) and add to \( K_s,t \) the edges \( u_1u_2, u_3u_4, \ldots, u_{s-1}u_s \).

20. Show that for every \( k \in \mathbb{N} \) the sequence \( (1, 1, 2, 2, \ldots, k, k) \) is graphic.

**Solution** We proceed by induction on \( k \). For \( k = 1 \) use \( P_2 \) that realizes \( (1, 1) \). So suppose that \( k \geq 1 \) and that \( (1, 1, 2, 2, \ldots, k, k) \) is graphic. I will show that \( (1, 1, 2, \ldots, k, k, k+1, k+1) \) is also graphic. Let \( G \) be a realization of \( (1, 1, 2, 2, \ldots, k, k) \) with vertices \( u_i \) and \( v_i \) having degree \( i \) for each \( i = 1, \ldots, k \). Add to \( G \) two new vertices \( w \) and \( w' \) and the edges \( wu_i \) and \( w'u_i \) for all \( i = 1, \ldots, k \). In the resulting graph \( w' \) and \( v_i \) have degree 1, \( u_1 \) and \( v_2 \) have degree 2, \( u_2 \) and \( v_3 \) have degree 3, \( u_{k-1} \) and \( v_k \) have degree \( k \), and \( u_k \) and \( w \) have degree \( k+1 \).