Exercise set 1 (Fundamental concepts)

1. Prove or disprove: The complement of a simple disconnected graph must be connected.

Solution The statement is true. Let G be a simple disconnected graph and $u, v \in V(G)$. If u and v belong to different components of G, then the edge $uv \in E(\bar{G})$. If u and v belong to the same component of G, choose a vertex w in another component of G. (G has at least two components, since it is disconnected.) But then the edges uw and wv belong to $E(\bar{G})$. That is, in all cases there is a u, v-path in $\bar{G}$. •

2. Prove that a bipartite graph has a unique bipartition (apart from interchanging the partite sets) if and only if it is connected.

Solution [if] Let G be a bipartite graph and choose $v \in V(G)$. Let X, Y and X', Y' be two different bipartitions of G with $v \in X$ and $v \in X'$. Since $X \neq X'$, at least one of $X \setminus X'$ and $X' \setminus X$ is non-empty. Also $Y \neq X'$, so both $Y \setminus X'$ and $X' \setminus Y$ can not be empty. Finally X, X', Y and Y' are independent sets. It follows that G is the disjoint union of two bipartite graphs, one with bipartition $X \setminus X', X' \setminus X$ and the other with bipartition $Y \setminus X', X' \setminus Y$. So G is not connected.

[only if] Let G be a disconnected bipartite graph, H a component of G and H' the disjoint union of all other components of G. H and H' are again bipartite, say, with bipartitions X, Y and X', Y' respectively. Then $X \cup X', Y \cup Y'$ and $X \cup Y', Y \cup X'$ are two different bipartitions of G. •

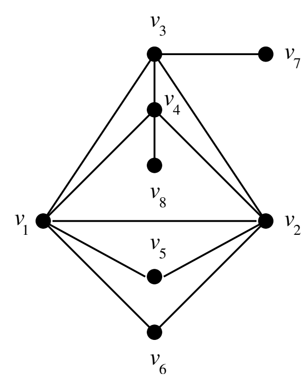
3. Prove or disprove: Every Eulerian bipartite graph contains an even number of edges.

Solution The statement is true. For the proof let G be an Eulerian bipartite graph with bipartition X, Y of its non-trivial component. We can count the number of edges in G as $e(G) = \sum_{v \in X} d(v)$. Since G is Eulerian, each degree $d(v)$ in this sum is even. •

4. Which of the following is a graphic sequence: $(5, 5, 5, 4, 2, 1, 1, 1)$ and $(5, 5, 4, 4, 2, 2, 1, 1)$? If it is graphic, produce a realization of the sequence, else prove why it is not graphic.

Solution $(5, 5, 5, 4, 2, 1, 1, 1)$ is graphic $\iff$ $(4, 4, 3, 1, 0, 1, 1)$, i.e., $(4, 4, 3, 1, 1, 1, 0)$ is graphic $\iff$ $(3, 2, 0, 0, 1, 0)$, i.e., $(3, 2, 1, 0, 0, 0)$ is graphic. The last sequence is not graphic, since a vertex of degree 3 in a simple graph must have three neighbors each of positive degree.

$(5, 5, 4, 4, 2, 2, 1, 1)$ is graphic $\iff$ $(4, 3, 3, 1, 1, 1, 1)$ is graphic $\iff$ $(2, 2, 0, 0, 1, 1)$, i.e., $(2, 2, 1, 1, 0, 0)$ is graphic $\iff$ $(1, 0, 1, 0, 0)$, i.e., $(1, 1, 0, 0, 0)$ is graphic $\iff$ $(0, 0, 0, 0)$ is graphic. But four isolated vertices realize $(0, 0, 0, 0)$ and so $(5, 5, 4, 4, 2, 2, 1, 1)$ is graphic. Starting from four isolated vertices one can work back and come up with the realization shown in the adjacent figure.



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5. Let n be a positive integer of the form $4k$ or $4k + 1$ for some $k \in \mathbb{N}$. Construct a simple graph G with n vertices, $n(n-1)/4$ edges and with $\Delta(G) - \delta(G) \leqslant 1$.

Solution [$n = 4k$] Let H be the complete bipartite graph with bipartition $\{u_1, \ldots, u_{2k}\}, \{v_1, \ldots, v_{2k}\}$. Take $G := H \setminus \{u_i v_i \mid i = 1, \ldots, k\}$.

[$n = 4k + 1$] Start with the graph G constructed for the case $n = 4k$. Add to G a new vertex w and the edges wu_i and wv_i for each $i = 1, \ldots, k$. The resulting graph is $2k$-regular. •

6. Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers.

Solution Consider the acquaintance graph G of a given set of six people with vertex set $\{v_1, \ldots, v_6\}$. We have to show that either G or $\bar{G}$ has a triangle. Since K_6 decomposes into G and $\bar{G}$, v_1 has a total of five assume that v_1 is joined to v_2, v_3, v_4 in G. If there is an edge $v_iv_j \in E(G)$ with $2 \leqslant i < j \leqslant 4$, then G contains the triangle with vertices v_1, v_i, v_j. On the other hand if no v_iv_j, $2 \leqslant i < j \leqslant 4$, belongs to G, they all belong to $\bar{G}$ and form a triangle in $\bar{G}$. •

7. Prove that a k-regular graph of girth 4 has at least $2k$ vertices.

Solution Let G be a k-regular graph of girth 4. Then G is simple (since loops and multiple edges produce 1-cycles and 2-cycles respectively). Choose any $u \in V(G)$ and let $N(u) = \{v_1, \ldots, v_k\}$. Since G has girth 4, any two v_i and v_j ($1 \leqslant i < j \leqslant k$) are non-adjacent. Thus v_1 has $k-1$ neighbors $w_1, \ldots, w_{k-1}$ not in the set $\{u, v_1, \ldots, v_k\}$. Thus we have already got $1 + k + (k-1) = 2k$ distinct vertices of G.
(**Remark:** If we also add the edges v_iw_j for all $i = 2, \ldots, k$ and $j = 1, \ldots, k-1$, we get a k-regular graph of girth 4 and with exactly $2k$ vertices. This is in fact the complete bipartite graph with bipartition $\{u, w_1, \ldots, w_{k-1}\}, \{v_1, \ldots, v_k\}$.) •

8. Prove that there exists a self-complementary graph with n vertices, if and only if n or $n-1$ is divisible by 4.

Solution [$n = 4t$] Let $X_1, \ldots, X_4$ be four pairwise disjoint sets each of cardinality t. Construct a graph G with $V(G) = \bigcup_{i=1}^{4} X_i$. $E(G)$ consists of the edges in the two complete graphs with vertex sets X_1 and X_4 respectively and the edges in the three bicliques (complete bigraphs) with bipartitions (X_1, X_2), (X_2, X_3) and (X_3, X_4) respectively. Then $\bar{G}$ has the edge set comprising the edges in the two complete graphs with vertex sets X_2 and X_3 respectively and the edges in the three bicliques with bipartitions (X_2, X_4), (X_4, X_1) and (X_1, X_3) respectively. It is (almost) immediate that $G \cong \bar{G}$.

[$n = 4t+1$] Construct the graph G on $4t$ vertices as described above. Add a new vertex $v \notin V(G)$ and the edges between v and every member of $X_1 \cup X_4$. The resulting graph G' is again self-complementary.

[$n = 4t+2$ or $n = 4t+3$] The total number of edges in K_n is odd in this case. So K_n can not decompose into a graph G and its complement $\bar{G}$ with $G \cong \bar{G}$. •

9. Argue that the Petersen graph has exactly 120 automorphisms.

Solution Let f be an automorphism of the Petersen graph G. Since f preserves adjacency, it maps the 5-cycle $(12, 34, 51, 23, 45)$ to a 5-cycle of G. With little care one can show that under the rules of adjacency in G every 5-cycle of G is of the form (ab, cd, ea, bc, de), where (a, b, c, d, e) is a permutation σ of $(1, 2, 3, 4, 5)$. Then it is an easy matter to check that every vertex ij of G is mapped by f to $\sigma(i)\sigma(j)$, i.e., f is determined by σ. Conversely each such permutation σ defines an automorphism of G. •

10. Let v be a cut-vertex of a simple graph G. Prove that $\bar{G} \setminus v$ is connected.

Solution First note that $\bar{G} \setminus v$ is the same as $\overline{G \setminus v}$. Now use Exercise 1. •

11. Let G_n be the simple graph with each vertex labeled by a permutation of $\{1, 2, \ldots, n\}$ and with two vertices adjacent if and only if their labels differ by an interchange of two adjacent entries. Prove that G_n is connected.

Solution Let $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_n)$ be two permutations of $\{1, 2, \ldots, n\}$. By a sequence of interchanges of adjacent entries one can convert $(i_1, i_2, \ldots, i_n)$ to $(1, 2, \ldots, n)$ — as in the bubble-sort algorithm. (This fact can be easily proved by induction on n.) Similarly a sequence of interchanges of adjacent entries produces $(1, 2, \ldots, n)$ from $(j_1, j_2, \ldots, j_n)$. If one reverses this second sequence of interchanges, one gets back $(j_1, j_2, \ldots, j_n)$ from $(1, 2, \ldots, n)$. •

12. Let G be a connected simple graph not containing P_4 or C_3 as an induced subgraph. Prove that G is a biclique (i.e., a complete bipartite graph).

Solution We first prove by induction on $k \in \mathbb{N}$ that G contains no cycles of length $2k+1$. Since G does not contain C_3 as (induced) subgraph, G does not contain 3-cycles. So consider $k \geqslant 2$ and suppose that G does not contain cycles of length $3, 5, \ldots, 2k-1$. Assume that G contains a $2k+1$-cycle Z. If G contains an edge e between non-adjacent vertices of Z, then G contains two smaller cycles Z_1 and Z_2. We choose e in such a way that the length of Z_1 is minimum. Now by the induction hypothesis neither Z_1 nor Z_2 is odd, subgraph of G. But then Z_1 (and hence G) contain P_4 as an induced subgraph, a contradiction.

Thus G contains no odd cycles, i.e., G is bipartite. Let X, Y be a bipartition of G. If $|X| = 1$ or $|Y| = 1$, the connectedness of G implies that G is complete. So consider $|X| > 1$ and $|Y| > 1$. Suppose that G is not complete, i.e., for some $x \in X$ and $y \in Y$ the edge $xy \notin E(G)$. Since G is connected, there is an x, y-path in G. Let P be a *minimal* x, y-path. Since x and y lie in different partite sets, the length of P is odd. Moreover, since $xy \notin E(G)$, the length of P is at least 3. Let x, y', x', y'' be the first four vertices on P (where $y' \neq y$, but one might have $y'' = y$). Since P is minimal, $xy'' \notin E(G)$. Also x and x' belong to the same partite set and hence $xx' \notin E(G)$. Similarly $y'y'' \notin E(G)$. It follows that the subgraph of G induced by $\{x, y, x', y''\}$ is (isomorphic to) the path P_4, a contradiction. •

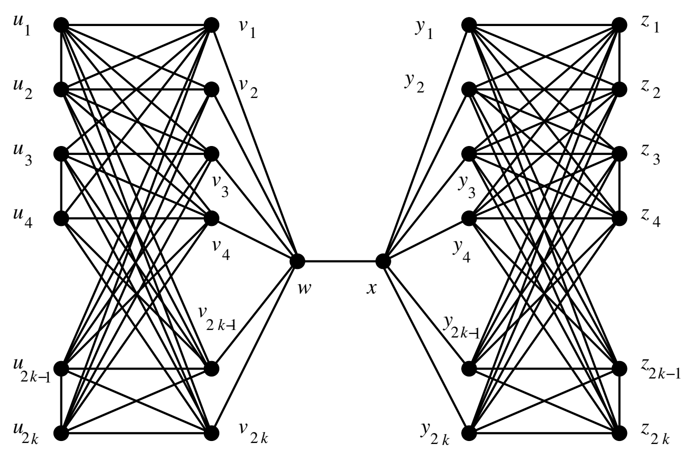
13. Let P and Q be paths of maximum length in a connected graph G. Prove that $V(P) \cap V(Q) \neq \emptyset$.

Solution Assume that $V(P) \cap V(Q) = \emptyset$. Let the endpoints of P be u and v and those of Q be x and y. Since G is connected, there is a u, x-path R in G. Since P and Q do not share a vertex, there is a portion S of R such that S is a p, q path with $p \in V(P)$ and $q \in V(Q)$ and such that S does not contain any other vertex of P or Q. Let $l := |P| = |Q|$. Either the u, p- or the p, v-subpath of P is of length $\geqslant l/2$. Because of symmetry we may assume that the former path, call it P', is not shorter than the latter. Similarly assume that the x, q-subpath Q' of Q is not shorter than the q, y-subpath of Q. That is, $|P'| \geqslant l/2$ and $|Q'| \geqslant l/2$. Now P' followed by S followed by Q' is a u, x-path of length $\geqslant (l/2) + 1 + (l/2) > l$, a contradiction to the maximality of $|P| = |Q|$. •

14. Prove that an even graph has no cut edge. For each $k \in \mathbb{N}$ produce a $2k+1$-regular simple graph with a cut edge.

Solution Let G be an even graph and let $e \in E(G)$ have endpoints u and v. Deleting e from G makes the degrees of u and v odd and leaves the degrees of other vertices unchanged (namely even). If u and v belong to different components in $G \setminus e$, then the component of $G \setminus e$ containing u would have exactly one odd vertex, a contradiction. So u and v are connected in $G \setminus e$. A u, v-path in $G \setminus e$ forms a cycle in conjunction with e, i.e., e lies on a cycle in G. Thus e is not a cut-edge.

The adjacent graph is $2k+1$-regular and simple with cut edge wx.



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15. Deduce that the total number of simple even graphs on a given set of n vertices is $2^{\binom{n-1}{2}}$.

Solution Let S be the set of all even simple graphs on the vertex set $\{v_1, \ldots, v_n\}$. Also let T be the set of all simple graphs on the vertex set $\{v_1, \ldots, v_{n-1}\}$. I will establish a bijection between S and T. Consider the map $f : S \to T$ defined by $G \mapsto G \setminus v_n$. For the converse consider $g : T \to S$ defined as follows. Take a graph $H \in T$ and let U denote the set of all odd vertices in H. Add to H the vertex v_n and an edge $v_i v_n$ for every $v_i \in U$. Since $|U|$ is even, the resulting graph ($\in S$) is even and is defined to be $g(H)$. It's easy to argue that $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$, i.e., f is a bijection. •

16. Argue that the Petersen graph has exactly 12 five-cycles.

Solution Call the Petersen graph G. First I claim that every edge l in G belongs to exactly four five cycles. For the proof let the endpoints of l be ab and cd for some $\{a, b, c, d\} \subseteq \{1, 2, 3, 4, 5\}$. (Since ab and

$\{a,b,c,d,e\} = \{1,2,3,4,5\}$. Similarly the three neighbors of cd are ab, ae and be. Therefore, every cycle of length $\geqslant 4$ containing the edge l must also contain one of the four paths having vertices xe, ab, cd, ye (in that order) for $x \in \{c,d\}$ and $y \in \{a,b\}$. For each such x and y the vertices xe and ye are non-adjacent and hence share a unique common neighbor, call it $v = v(x,y)$. Then (v, xe, ab, cd, ye) is a 5-cycle for each x, y. This proves the claim.

Let n be the number of 5-cycles in G. Summing up the edges in all these cycles gives a total of $5n$ edges. In this sum each edge of G is counted four times in view of the previous claim. Since G has 15 edges, we have $5n = 4 \times 15 = 60$, i.e., $n = 12$. •

17. Let G be an n-vertex simple graph with $n \geqslant 2$. Determine the maximum possible number of edges in G under each of the following conditions:
i) G has an independent set of size k.
ii) G has exactly k components.
iii) G is disconnected.

Solution i) $\binom{n}{2} - \binom{k}{2}$. Achieved by the graph obtained by deleting from K_n all edges with both endpoints in a chosen k-element subset of $V(K_n)$.

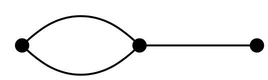
ii) $\binom{n-k+1}{2}$. Let $H_1, \ldots, H_k$ be the k components of G with $n_i := n(H_i)$ for $i = 1, \ldots, k$. We may assume (after rearranging, if necessary) that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$. Then $e(G) = e(H_1) + e(H_2) + \cdots + e(H_k) \leqslant \binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2} = (n_1^2 + n_2^2 + \cdots + n_k^2 - n)/2$. This number of edges corresponds to that of a disjoint union of k complete graphs $(K_{n_1} + K_{n_2} + \cdots + K_{n_k})$. Now for $n_i > 1$ we have $e(K_{n_1+1} + K_{n_2} + \cdots + K_{n_{i-1}} + K_{n_i-1} + K_{n_{i+1}} + \cdots + K_{n_k}) - e(K_{n_1} + K_{n_2} + \cdots + K_{n_k}) = (n_1 - n_i) + 1 \geqslant 1$, i.e., we gain at least an edge by pushing a vertex from the i-th component to the first one. Thus if $n_2 = \cdots = n_k = 1$ (so that $n_1 = n - k + 1$), we have the maximum value of $e(G)$, i.e., $e(G) \leqslant \binom{n-k+1}{2}$. This bound is achieved by $K_{n-k+1} + K_1 + \cdots + K_1$.

iii) $\binom{n-1}{2}$. A disconnected graph G has $k \geqslant 2$ components and hence by Part ii) can have a maximum of $\binom{n-k+1}{2}$ edges. This number is maximized for $k = 2$, which implies that $n(G) \leqslant \binom{n-1}{2}$. This bound is achieved by $K_{n-1} + K_1$. •

18. Show that every simple graph with at least two vertices contains at least two vertices of the same degree. Does the result continue to hold if 'simple' is replaced by 'loopless'?

Solution Let $n \geqslant 2$ and G a simple graph with n vertices. Since the vertex degrees can assume only n possible values $0, 1, \ldots, n-1$, the n vertices of G have pairwise distinct degrees if and only if G has a vertex v_i of degree i for each $i = 0, \ldots, n-1$. But then v_{n-1} is adjacent to every other vertex in $V(G)$ and in particular to v_0, a contradiction.

The result does not necessarily hold for loopless graphs. For example consider the adjacent 3-vertex loopless graph with the degree sequence $(3, 2, 1)$.



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19. Let n, s, t be non-negative integers with $n = s + t > 0$. Find necessary and sufficient conditions on n, s, t such that there exists a connected simple n-vertex graph with s vertices of odd degree and t vertices of even degree.

Solution The necessary and sufficient condition sought for is that s be even. The necessity is an immediate consequence of the degree-sum formula. For the converse let us be given an even s and any t. If $s = 0$, then consider the cycle C_t. If $t = 0$, consider the complete graph K_s. Finally suppose that $s > 0$ and $t > 0$ and start with the complete biclique $K_{s,t}$. If t is odd, $K_{s,t}$ is a desired graph. Otherwise let $X = \{u_1, \ldots, u_s\}, Y = \{v_1, \ldots, v_t\}$ be the bipartition of the vertices of $K_{s,t}$ and add to $K_{s,t}$ the edges $u_1u_2, u_3u_4, \ldots, u_{s-1}u_s$. •

20. Show that for every $k \in \mathbb{N}$ the sequence $(1, 1, 2, 2, \ldots, k, k)$ is graphic.

Solution We proceed by induction on k. For $k = 1$ consider P_2 that realizes $(1, 1)$. So suppose that $k \geqslant 1$ and that $(1, 1, 2, 2, \ldots, k, k)$ is graphic. I will show that $(1, 1, 2, 2, \ldots, k, k, k+1, k+1)$ is also graphic. Let G be a realization of $(1, 1, 2, 2, \ldots, k, k)$ with vertices u_i and v_i having degree i for each $i = 1, \ldots, k$. Add to G two new vertices w and w' and the edges ww' and wu_i for all $i = 1, \ldots, k$. In the resulting graph w' and v_1 have degree 1, u_1 and v_2 have degree 2, u_2 and v_3 have degree 3, $\ldots$, u_{k-1} and v_k have degree k, and u_k and w have degree $k+1$. •