Mid Semester Examination, Spring 2002-03

Total points: 30 (+ 6 bonus) February 26, 2003 Total time: 2 hours

[*This test is closed-book, but open-notes.*]

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1. Which of the following propositions are true? Provide brief explanations to justify your verdicts. **(2 × 6)**

(a) If u and v are the only vertices of odd degree in a graph G, then G contains a u,v-path.

(b) Let G be a connected graph with at least two vertices and with $\delta(G) < \Delta(G)$. Deleting a vertex of degree $\delta(G)$ can not reduce the average degree.

(c) The edge uv in a simple graph G is a cut-edge, if and only if $n(G) \geqslant d(u) + d(v)$.

(d) Every graph with fewer edges than vertices has a component that is a tree.

(e) If G is an Eulerian graph with edges e, e' sharing a vertex, then G has an Eulerian circuit in which e and e' appear consecutively.

(f) Let $D = (a, a, \ldots, a, b, b, \ldots, b)$ be a sequence of positive integers with $k > 0$ occurrences of a and $l > 0$ occurrences of b. Also assume that $ka + lb$ is even and that $0 < b < a < k + l$. Then D is a graphic sequence.

2. Let G_1 and G_2 be simple graphs with $n(G_i) = n_i$ and $e(G_i) = e_i$ for $i = 1, 2$. The product $G_1 \times G_2$ is defined as the graph with vertex set $V(G_1) \times V(G_2)$ and with (u_1, u_2) and (v_1, v_2) adjacent, if and only if either

$u_1 = v_1$ and u_2 is adjacent to v_2 in G_2

or

$u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

(a) Draw $P_2 \times P_3$. **(1)**

(b) Prove that $n(G_1 \times G_2) = n_1 n_2$ and $e(G_1 \times G_2) = n_1 e_2 + n_2 e_1$. **(1 + 2)**

(c) Prove or disprove: If G_1 and G_2 are regular, then so is $G_1 \times G_2$. **(2)**

3. Let G be a connected graph with at least three vertices. Prove that G has two vertices x, y such that:

1) $G \setminus \{x, y\}$ is connected, and

2) x, y are adjacent or have a common neighbor (in G).

(**Hint:** Consider a longest path in G.) **(6)**

4. Let G be an n-vertex simple graph with the property that for some k, $1 < k < n-1$, every k-vertex induced subgraph of G has m edges.

(a) Show that for $k \leqslant l \leqslant n$ every l-vertex induced subgraph of G has $m\binom{l}{k}/\binom{l-2}{k-2}$ edges. **(3)**

(b) Deduce that G is either K_n or $\bar{K}_n$. (**Hint:** Use Part (a) to conclude that the number of edges between u and v is independent of the choice of $u, v \in V(G)$.) **(3)**

5. **(a)** Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1. **(3)**

(b) Show that the bound of Part (a) is best possible by constructing an n-vertex tree with exactly Δ vertices of degree 1 for every choice of n, Δ with $n > \Delta \geqslant 2$. **(3)**

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