Solutions to the exercises in the Mid Semester Examination, Spring 2002-03

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1. Which of the following propositions are true? Provide brief explanations to justify your verdicts.

(a) If u and v are the only vertices of odd degree in a graph G, then G contains a u,v-path.

(b) Let G be a connected graph with at least two vertices and with $\delta(G) < \Delta(G)$. Deleting a vertex of degree $\delta(G)$ can not reduce the average degree.

(c) The edge uv in a simple graph G is a cut-edge, if and only if $n(G) \geqslant d(u) + d(v)$.

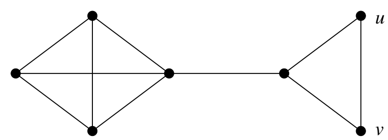
(d) Every graph with fewer edges than vertices has a component that is a tree.

(e) If G is an Eulerian graph with edges e, e' sharing a vertex, then G has an Eulerian circuit in which e and e' appear consecutively.

(f) Let $D = (a, a, \ldots, a, b, b, \ldots, b)$ be a sequence of positive integers with $k > 0$ occurrences of a and $l > 0$ occurrences of b. Also assume that $ka + lb$ is even and that $0 < b < a < k + l$. Then D is a graphic sequence.

Solution **(a) True:** Let H be the component of G, that contains u. If $v \notin V(H)$, then H contains an odd number (one) of vertices of odd degree, a contradiction. Thus u and v lie in the same component of G.

(b) False: Consider the following graph:

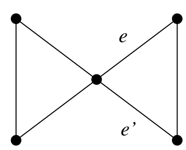


The average degree of G is $2e(G)/n(G) = 20/7 = 2.857\ldots$, whereas the average degree of $G \setminus v$ is $2e(G \setminus v)/n(G \setminus v) = 16/6 = 2.666\ldots$.

(c) False: Again consider the graph of Part (b). uv is not a cut-edge (since it lies on a cycle), whereas $7 = n(G) > d(u) + d(v) = 2 + 2$. The 'only if' part is however true, as one can prove (easily).

(d) True: Let G be a graph with $e(G) < n(G)$ and let $G_1, \ldots, G_k$ be the components of G. Assume that no G_i is a tree. Then $e(G_i) \geqslant n(G_i)$ for all $i = 1, \ldots, k$ (because a connected graph H with $e(H) < n(H) - 1$ is not connected, with $e(H) = n(H) - 1$ is a tree and with $e(H) \geqslant n(H)$ is not acyclic). Summing up for all $i = 1, \ldots, k$ gives $e(G) \geqslant n(G)$, a contradiction.

(e) False: As a counterexample consider the following graph:



(f) False: The sequence $(3, 3, 1, 1)$ is not graphic, for if it were so, then $(2, 0, 0)$ would also be graphic, but a simple graph having a vertex of degree two must contain at least two other vertices of positive degree. •

2. Let G_1 and G_2 be simple graphs with $n(G_i) = n_i$ and $e(G_i) = e_i$ for $i = 1, 2$. The product $G_1 \times G_2$ is defined as the graph with vertex set $V(G_1) \times V(G_2)$ and with (u_1, u_2) and (v_1, v_2) adjacent, if and only if either

$u_1 = v_1$ and u_2 is adjacent to v_2 in G_2

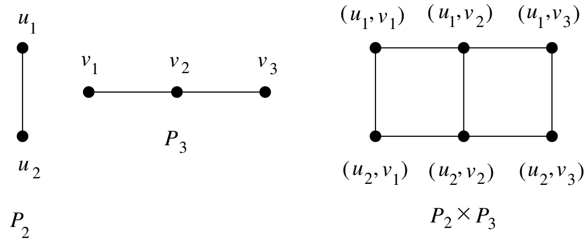
or

$u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

(a) Draw $P_2 \times P_3$.

(b) Prove that $n(G_1 \times G_2) = n_1 n_2$ and $e(G_1 \times G_2) = n_1 e_2 + n_2 e_1$.

(c) Prove or disprove: If G_1 and G_2 are regular, then so is $G_1 \times G_2$.



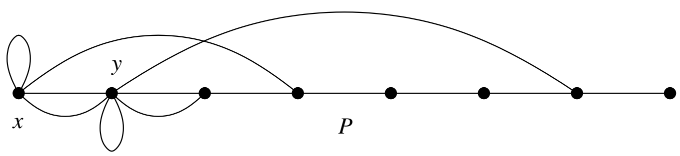
(b) Let us denote $G := G_1 \times G_2$. Since $V(G) = V(G_1) \times V(G_2)$, we have $n(G) = n_1 n_2$. In order to count the number of edges in G first note that since G is simple, the neighborhood $N_G(u,v)$ of (u,v) in G is the disjoint union of the sets $\{(u',v) \mid u' \in N_{G_1}(u)\}$ and $\{(u,v') \mid v' \in N_{G_2}(v)\}$. Therefore, $d_G(u,v) = d_{G_1}(u) + d_{G_2}(v)$. Summing over all (u,v) pairs yields $2e(G) = \sum_{u,v} d_G(u,v) = n_2 \sum_u d_{G_1}(u) + n_1 \sum_v d_{G_2}(v) = n_2 \times 2e_1 + n_1 \times 2e_2$, i.e., $e(G) = n_1 e_2 + n_2 e_1$.

(c) The statement is true. Let G_1 be k_1-regular and let G_2 be k_2-regular. We have deduced in Part (b) that for $(u,v) \in V(G)$ we have $d_G(u,v) = d_{G_1}(u) + d_{G_2}(v) = k_1 + k_2$, which is independent of the choice of u and v. Thus G is $k_1 + k_2$-regular. •

3. Let G be a connected graph with at least three vertices. Prove that G has two vertices x, y such that:
1) $G \setminus \{x,y\}$ is connected, and
2) x, y are adjacent or have a common neighbor (in G).
(**Hint:** Consider a longest path in G.)

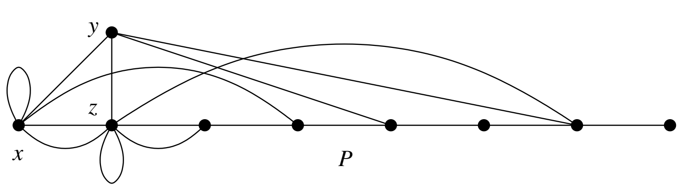
Solution Let P be a path of maximum length in G. Since G is connected and has at least three vertices, it follows that P contains at least three points. Let x be an endpoint of P. By the maximality of P any neighbor of x lies on P. Let z be the neighbor of x along P. We consider two separate cases.

Case 1: z does not have a neighbor outside P.
In this case take $y := z$, $G' := G \setminus \{x,y\}$. The situation is described in the following figure:



Let $u, v \in V(G') = V(G) \setminus \{x,y\}$. If u, v are both on P, then there is a u,v-path (a subpath of P) in G'. Now consider the case that one of u and v is on P, the other outside P. By symmetry one may take u outside P. Since G is connected, there is a u,v-path Q in G. Let w be the first vertex on Q, that belongs to P. Since neither x nor y has a neighbor outside $V(P)$, it follows that $w \notin \{x,y\}$. Thus the part of Q from u to w and the part of P from w to v form a u,v-path in G'. Finally consider the case that both u and v are outside P. There is a u,v-path in G, call it Q again. If Q does not contain a vertex of P, then Q remains in G'. Otherwise let w and w' be the first and last vertices of Q, that lie on P. We have $w, w' \notin \{x,y\}$ as before. Now the part of Q from u to w, the part of P from w to w' and the part of Q from w' to v produce a u,v-path in G'.

Case 2: z has a neighbor y outside P.
Take $G' := G \setminus \{x,y\}$. The situation is now described in the following figure. Note that y is not necessarily uniquely determined in this case, but any choice y of a neighbor of z outside P will serve our purpose.



Since P is a path of maximum length in G, it follows that y (like x) has all its neighbors (except perhaps itself) on P. Take any $u, v \in V(G') = V(G) \setminus \{x,y\}$. If u, v are both on P, then there is a u,v-path (a in G. Then $w \neq x$, since x has no neighbor outside P. The similar property of y implies that the part of Q from u to w can not pass through y. But then the u, w-subpath of Q and the w, v-subpath of P constitute a u, v-path in G'. Finally suppose that both u and v are outside P. Consider a u, v-path Q in G. If Q does not meet P, it does not contain y as well, i.e., Q remains in G'. Otherwise take w and w' to be the first and last vertices of Q on P. As before neither w nor w' equals x or y and the u, w-subpath of Q, the w, w'-subpath of P and the w', v-subpath of Q constitute a u, v-path in G'. •

4. Let G be an n-vertex simple graph with the property that for some k, $1 < k < n-1$, every k-vertex induced subgraph of G has m edges.

(a) Show that for $k \leqslant l \leqslant n$ every l-vertex induced subgraph of G has $m\binom{l}{k}/\binom{l-2}{k-2}$ edges.

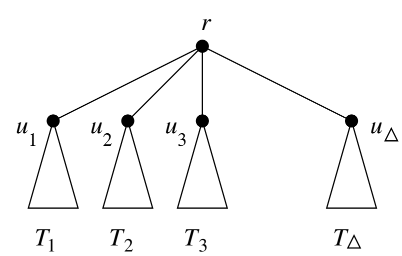
(b) Deduce that G is either K_n or $\bar{K}_n$. (**Hint:** Use Part (a) to conclude that the number of edges between u and v is independent of the choice of $u, v \in V(G)$.)

Solution **(a)** Let H be an l-vertex induced subgraph of G. For any k-subset X of $V(H)$ the subgraph of H (i.e., of G) induced by X has m edges by hypothesis. If we vary X over all of the $\binom{l}{k}$ possibilities (of choosing a k-subset from an l-set) and sum the numbers of edges in all these k-vertex induced subgraphs of H, we get a total of $m\binom{l}{k}$ edges. All these edges are those of H counted multiple times. An edge $uv \in E(H)$ is counted in $H[X]$ if and only if both u and v belong to X. But the number of k-subsets of $V(H)$ containing u and v is $\binom{l-2}{k-2}$. (Pick the $k-2$ vertices of $X \setminus \{u, v\}$ from $V(H) \setminus \{u, v\}$ in any possible way.) Thus in the sum of numbers of edges built above every edge uv of H is counted exactly $\binom{l-2}{k-2}$ times. It follows that $e(H) = m\binom{l}{k}/\binom{l-2}{k-2}$.

(b) Let u, v be arbitrary (but distinct) vertices of G and let $\epsilon(u, v) \in \{0, 1\}$ denote the number of edges in G between u and v. For $x \in V(G)$ let $n_1(x)$ denote the number of edges of G having x as one endpoint. Also for distinct $x, y \in V(G)$ let $n_2(x, y)$ denote the number of edges in G having x or y (or both) as one endpoint. It then follows that $\epsilon(u, v) = n_1(u) + n_1(v) - n_2(u, v)$. Now $n_1(u) = n(G) - n(G \setminus u)$ and hence by Part (a) we have $n_1(u) = m\left[\binom{n}{k}/\binom{n-2}{k-2} - \binom{n-1}{k}/\binom{n-3}{k-2}\right]$. Similarly, $n_1(v) = m\left[\binom{n}{k}/\binom{n-2}{k-2} - \binom{n-1}{k}/\binom{n-3}{k-2}\right]$. Finally, $n_2(u, v) = n(G) - n(G \setminus \{u, v\}) = m\left[\binom{n}{k}/\binom{n-2}{k-2} - \binom{n-2}{k}/\binom{n-4}{k-2}\right]$. Therefore, $\epsilon(u, v) = m\left[\binom{n}{k}/\binom{n-2}{k-2} - 2\binom{n-1}{k}/\binom{n-3}{k-2} + \binom{n-2}{k}/\binom{n-4}{k-2}\right]$. This quantity is independent of the choice of u and v. If $\epsilon(u, v) = 0$ for all $u, v \in V(G)$, then $G = \bar{K}_n$. On the other hand, if $\epsilon(u, v) = 1$ for all $u, v \in V(G)$, then $G = K_n$. •

5. **(a)** Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1.

(b) Show that the bound of Part (a) is best possible by constructing an n-vertex tree with exactly Δ vertices of degree 1 for every choice of n, Δ with $n > \Delta \geqslant 2$.



Solution **(a)** Let T be a tree with $\Delta(T) = \Delta > 1$. We view T as a tree rooted at a vertex r of degree Δ. Let $u_1, \ldots, u_\Delta$ be the children of r and let T_i be the subtree of T rooted at u_i (for each $i = 1, \ldots, \Delta$). If T_i has no vertex other than u_i, then u_i is a vertex of T of degree 1. On the other hand, if T_i has $\geqslant 2$ vertices, it has at least 2 vertices of degree 1 (of which one can be u_i itself), that is, at least one vertex in $V(T_i) \setminus \{u_i\}$ is a vertex of degree 1 in T_i and hence in T as well.

(b) Take $T_2, \ldots, T_\Delta$ to be single-vertex trees and T_1 to be a path with $n - \Delta$ vertices and with u_1 as an endpoint. •