Exercise set 2

Note: The solutions of the exercises marked by single stars are to be submitted by each student in Group I. The solutions of the exercises marked by double stars are to be submitted by each student in Group II.

Trees

1. Let T, T' be two spanning trees of a connected graph G. For an edge $e \in E(T) \setminus E(T')$ prove that there is an edge $e' \in E(T') \setminus E(T)$ such that both $T' + e - e'$ and $T - e + e'$ are spanning trees of G.

2. Let G be a connected graph and $e \in E(G)$. Prove that:

(a) e is a cut-edge if and only if e belongs to every spanning tree.

(b) e is a loop if and only if e belongs to no spanning tree.

3. Using Cayley's formula prove that the graph obtained from K_n by deleting an edge has exactly $(n-2)n^{n-3}$ spanning trees.

4. Use the matrix tree theorem to compute $\tau(K_{s,t})$.

* **5.** Determine the graph(s) G with $n \geqslant 3$ vertices with the property that $G \setminus u$ is a tree for every $u \in V(G)$.

** **6.** **(a)** Prove that for any $u, v, w \in V(G)$ we have the triangle inequality: $d(u, w) \leqslant d(u, v) + d(v, w)$.

(b) Show that $\operatorname{diam} G \leqslant 2 \operatorname{rad} G$ for every graph G.

(c) For positive integers r and d satisfying $r \leqslant d \leqslant 2r$ construct a simple graph G with $\operatorname{rad} G = r$ and $\operatorname{diam} G = d$.

Matchings and factors

7. **(a)** Characterize all simple graphs G for which $\alpha(G) = 1$.

(b) Characterize all simple graphs G for which $\alpha'(G) = 1$.

(c) Characterize all simple graphs G for which $\beta(G) = 1$.

(d) Characterize all simple graphs G for which $\beta'(G) = 1$.

8. **(a)** Prove that a bipartite graph G has a perfect matching if and only if $|N(S)| \geqslant |S|$ for every $S \subseteq V(G)$.

(b) Present an infinite class of examples to prove that this characterization does not hold for all graphs.

9. Let G be a non-trivial simple graph. Prove that $\alpha(G) \leqslant n(G) - e(G)/\Delta(G)$. Conclude that $\alpha(G) \leqslant n(G)/2$ when G is also regular.

10. For every graph G prove that $\beta(G) \leqslant 2\alpha'(G)$. For each $k \in \mathbb{N}$ construct a simple graph G with $\alpha'(G) = k$ and $\beta(G) = 2k$.

* **11.** Prove or disprove: Every tree has at most one perfect matching.

** **12.** Prove that a tree T has a perfect matching, if and only if $o(T \setminus v) = 1$ for every $v \in V(T)$.

* **13.** Let G be a 3-regular graph with at most two cut edges. Prove that G has a 1-factor.

Cuts and connectivity

14. Give a counterexample to the following statement and add a hypothesis to correct it: If e is a cut-edge in G, then at least one endpoint of e is a cut vertex.

15. Let G be the Petersen graph. Show that $\kappa(G) = \kappa'(G) = 3$.

16. Prove that the symmetric difference of two different edge cuts is again an edge cut.

17. Prove or disprove: If P is a u, v-path in a 2-connected graph G, then there is a u, v-path Q internally disjoint from P.

** **18.** Let G be a simple graph with at least three vertices. Prove that G is 2-connected if and only if for every triple (x, y, z) of distinct vertices G has an x, y-path through z.

* **19.** For each choice of integers κ, κ', δ with $0 < \kappa \leqslant \kappa' \leqslant \delta$ construct a simple graph G having $\kappa(G) = \kappa$, $\kappa'(G) = \kappa'$ and $\delta(G) = \delta$.

** **20.** Prove that the block-cutpoint graph of a connected graph G is a tree whose leaves are (vertices representing) blocks of G.

Planarity

21. Let G be a plane graph. Show that $(G^*)^* = G$ if and only if G is connected.

22. For $n \geqslant 2$ determine the maximum number of edges in a simple outerplane graph with n vertices.

23. Use Kuratowski's theorem to prove that G is outerplanar if and only if G contains no subdivision of K_4 or $K_{2,3}$.

* **24.** **(a)** Let G be an n-vertex simple planar graph of girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges.

(b) Use Part (a) to prove that the Petersen graph is non-planar.

** **25.** **(a)** Prove that every n-vertex plane graph isomorphic to its dual has exactly $2n - 2$ edges.

(b) For $n \geqslant 4$ construct a simple n-vertex plane graph isomorphic to its dual.