Exercise set 1 (Fundamental concepts)

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1. Prove or disprove: The complement of a simple disconnected graph must be connected.

2. Prove that a bipartite graph has a unique bipartition (apart from interchanging the partite sets) if and only if it is connected.

3. Prove or disprove: Every Eulerian bipartite graph contains an even number of edges.

4. Which of the following is a graphic sequence: $(5, 5, 5, 4, 2, 1, 1, 1)$ and $(5, 5, 4, 4, 2, 2, 1, 1)$? If it is graphic, produce a realization of the sequence, else prove why it is not graphic.

5. Let n be a positive integer of the form $4k$ or $4k + 1$ for some $k \in \mathbb{N}$. Construct a simple graph G with n vertices, $n(n-1)/4$ edges and with $\Delta(G) - \delta(G) \leqslant 1$.

6. Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers.

7. Prove that a k-regular graph of girth 4 has at least $2k$ vertices.

8. Prove that there exists a self-complementary graph with n vertices, if and only if n or $n-1$ is divisible by 4.

9. Argue that the Petersen graph has exactly 120 automorphisms.

10. Let v be a cut-vertex of a simple graph G. Prove that $\bar{G} \setminus v$ is connected.

11. Let G_n be the simple graph with each vertex labeled by a permutation of $\{1, 2, \ldots, n\}$ and with two vertices adjacent if and only if their labels differ by an interchange of two adjacent entries. Prove that G_n is connected.

12. Let G be a connected simple graph not containing P_4 or C_3 as an induced subgraph. Prove that G is a biclique (i.e., a complete bipartite graph).

13. Let P and Q be paths of maximum length in a connected graph G. Prove that $V(P) \cap V(Q) \neq \emptyset$.

14. Prove that an even graph has no cut edge. For each $k \in \mathbb{N}$ produce a $2k+1$-regular simple graph with a cut edge.

15. Deduce that the total number of simple even graphs on a given set of n vertices is $2^{\binom{n-1}{2}}$.

16. Argue that the Petersen graph has exactly 12 five-cycles.

17. Let G be an n-vertex simple graph with $n \geqslant 2$. Determine the maximum possible number of edges in G under each of the following conditions:
i) G has an independent set of size k.
ii) G has exactly k components.
iii) G is disconnected.

18. Show that every simple graph with at least two vertices contains at least two vertices of the same degree. Does the result continue to hold if ‘simple’ is replaced by ‘loopless’?

19. Let n, s, t be non-negative integers with $n = s + t > 0$. Find necessary and sufficient conditions on n, s, t such that there exists a connected simple n-vertex graph with s vertices of odd degree and t vertices of even degree.

20. Show that for every $k \in \mathbb{N}$ the sequence $(1, 1, 2, 2, \ldots, k, k)$ is graphic.