17622 Advanced graph theory

End Semester Examination, Spring 2002-03

Total points: 60 | April 28, 2003 | Total time: 3 hours

[*This test is open-West (and open-notes), but closed from all other directions.*]

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1. Which of the following statements are true? Give brief explanations. **(2 × 8)**

(a) A simple graph having no cycles has exactly one spanning tree.

(b) An M-alternating path of odd length must be M-augmenting.

Graph theory is damn easy!

(c) A graph G with at least three vertices is 2-connected if and only if each edge of G lies on some cycle (not necessarily unique).

(d) A 1-edge-connected graph must contain at least one cut edge.

(e) $\kappa(G) = \kappa'(G) = \delta(G)$ for a 3-regular graph G.

(f) All cut edges in a plane graph belong to the boundary of the outer face.

(g) Every subgraph of a non-planar graph, induced by six or more vertices, is non-planar.

(h) Let G be a plane graph containing a cut vertex. Then the dual G^* has a cut vertex.

2. Prove or disprove: **(2 × 5)**

(a) G is a forest if and only if every connected subgraph of G is an induced subgraph.

(b) Let G be a connected graph with at least two vertices and let $v \in V(G)$. Then there is a maximum matching of G that saturates v.

What? You call it difficult!

(c) If a graph G has a perfect matching, then $|N(S)| \geqslant |S|$ for every $S \subseteq V(G)$.

(d) A simple connected graph G with $n(G) = e(G)$ is 3-colorable.

(e) Every non-trivial connected graph has two vertices that are not cut vertices.

3. Count the number of spanning cycles (also called *Hamiltonian circuits*) in: **(2 × 3)**

I am good at filthy counting.

(a) K_n for $n \geqslant 3$.

(b) $K_n \setminus e$ for $n \geqslant 3$, where $e \in E(K_n)$.

(c) $K_{n,n}$ for $n \geqslant 2$.

4. Let G be a bipartite graph. Prove that $\alpha(G) = n(G)/2$, if and only if G has a perfect matching. **(4)**

5. Let G be a graph with exactly k components. Let $b(G)$ denote the number of blocks in G and for $v \in V(G)$ let $b(v)$ denote the number of blocks of G to which v belongs. Derive the formula: $b(G) - k = \sum_{v \in V(G)} [\, b(v) - 1\,]$. **(4)**

Hmmm...! So these are your big guns.

6. Let G be a simple n-vertex graph. Prove that: **(4 × 2)**

(a) If $\delta(G) \geqslant n - 2$, then $\kappa(G) = \delta(G)$.

(b) If $\delta(G) \geqslant \lfloor n/2 \rfloor$, then $\kappa'(G) = \delta(G)$.

7. Let $n \in \mathbb{N}$, $n \geqslant 3$, and let G be the simple graph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{v_i v_j \mid |i - j| \leqslant 3\}$. Prove that G is a maximal planar graph. **(4)**

Wow!!!

8. Let G be a (connected) graph having spanning trees of diameters 2 and l (for some $l \geqslant 2$). Show that for every k, $2 \leqslant k \leqslant l$, G has a spanning tree of diameter k. **(8)**