Solutions to the exercises in the End Semester Examination, Spring 2002-03

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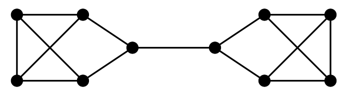
1. (a) False: A *disconnected* simple acyclic graph does not have a spanning tree.

(b) False: An M-augmenting path must have the endpoints M-unsaturated. If the first and last edges on an odd M-alternating path are in M, this condition is not satisfied.

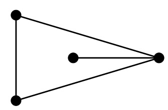
(c) False: The bow-tie has a cut-vertex but no cut-edges.

(d) False: A 1-edge-connected graph can also be 2-edge-connected.

(e) False: For a 3-connected graph G one has $\kappa(G) = \kappa'(G)$. One may have this common value less than $\delta(G)$. For example, the following graph is 3-regular with $\kappa(G) = \kappa'(G) = 1$.



(f) False: Consider the following plane graph:



(g) False: C_6 is a planar graph, but is an induced subgraph of the non-planar graph $C_6 + K_5$.

(h) False: Consider the plane graph G of Part (f). G has a cut vertex (the vertex of degree 3). G^* has two vertices and hence cannot have a cut vertex.

2. (a) True: Let G be a forest and let H be a connected subgraph of G. Since G is acyclic, so is H, i.e., H is a tree. Let e be an edge in $E(G) \setminus E(H)$ and with endpoints in $V(H)$. Adding e to H yields a subgraph of G containing a cycle, a contradiction. So there are no such edges e, i.e., H is an induced subgraph of G.

For the converse consider a spanning tree T of a component H of G. Since T is an induced subgraph, we have $H = T$, i.e., every component of G is a tree.

(b) True: Let $u \neq v$ be a neighbor of v in G. (Such a neighbor exists, since G is connected and has at least two vertices.) Start with the matching M consisting of an edge with endpoints u and v. M saturates u and v. If M is already maximum, we are done, else choose an M-augmenting path and get a bigger matching (with one more edge and with two more saturated vertices). If the augmented matching is maximum, we are through, else augment it further until a maximum matching is obtained. Each such augmentation leaves v in the set of saturated vertices.

(c) True: Let $\{e_1, \ldots, e_k\}$ be a perfect matching of G and let u_i and v_i be the endpoints of e_i. Then $V(G)$ consists of the $2k$ vertices $u_1, \ldots, u_k, v_1, \ldots, v_k$. Let $S = \{u_{i_1}, \ldots, u_{i_s}\} \cup \{v_{j_1}, \ldots, v_{j_t}\}$ be a subset of $V(G)$ of size $s+t$. Then the neighborhood of S contains (at least) the $s+t$ vertices $u_{j_1}, \ldots, u_{j_t}, v_{i_1}, \ldots, v_{i_s}$, i.e., $|N(S)| \geqslant |S|$.

(d) True: G is a tree plus an extra edge, i.e, G contains a unique cycle. Let e be an edge on this cycle. $G \setminus e$ is a tree and hence bipartite and consequently 2-colorable. Consider a proper 2-coloring of $G \setminus e$. If the two endpoints of e receive different colors, G also is 2-colorable. Otherwise change the color of (exactly) one of the endpoints of e to a third color to get a proper 3-coloring of G.

(e) True: Choose a pair u, v of (distinct) vertices in G such that the distance $d(u, v)$ is maximum. Suppose that u is a cut-vertex of G, i.e., $G \setminus u$ is disconnected. Choose a vertex w from a component other than the one containing v. Every v, w-path goes through u, i.e., $d(v, w) > d(v, u)$, a contradiction to the choice of u, v. Thus u is not a cut-vertex of G. Similarly v is not a cut-vertex of G.

3. (a) Let $V(K_n) = \{1, 2, \ldots, n\}$. A spanning cycle in K_n can be traversed starting from the vertex with label 1 and ending in the same vertex, i.e., a spanning cycle of K_n is of the form $(1, v_2, \ldots, v_n)$ where $v_2, \ldots, v_n$ is a permutation of $2, \ldots .n$. There $(n-1)!$ such permutations. Two different permutations $v_2, \ldots, v_n$ and

$(n-1)!/2$ spanning cycles.

(b) The number of spanning cycles in $K_n \setminus e$ is $(n-1)!/2 - k$, where k is the number of spanning cycles of K_n containing the edge e. By symmetry k is independent of the choice of e. Letting S denote the set of all spanning cycles of K_n then yields $\sum_{C \in S} e(C) = n \times (n-1)!/2 = k \times n(n-1)/2$, i.e., $k = (n-2)!$. $K_n \setminus e$ has exactly $(n-1)!/2 - (n-2)! = (n-3)(n-2)!/2$ spanning cycles.

(c) Every spanning cycle of $K_{n,n}$ alternates between the vertices of the two partite sets $X = \{1, 2, \ldots, n\}$ and $Y = \{1', 2', \ldots, n'\}$ of $K_{n,n}$ and is of the form $(1, v_1, u_2, v_2, \ldots, u_n, v_n)$, where $u_2, \ldots, u_n$ is a permutation of $2, \ldots, n$ and $v_1, \ldots, v_n$ is a permutation of $1', 2', \ldots, n'$. There are $(n-1)!n!$ such choices. Finally taking into account the two possible directions of traversal along a given cycle we conclude that $K_{n,n}$ has exactly $(n-1)!n!/2$ spanning cycles.

4. [if] First note that a graph with isolated vertices cannot have a perfect matching; so G contains no isolated vertices. We have $\alpha'(G) = n(G)/2$. But $\alpha'(G) + \beta'(G) = n$, so that $\beta'(G) = n(G)/2$. Finally since G is a bipartite graph without isolated vertices, we have $\alpha(G) = \beta'(G)$.

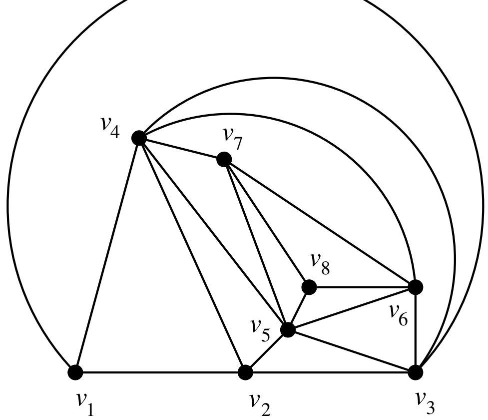
[only if] Let G be an X, Y-bigraph. Since $\alpha(G) = n(G)/2$, $|X| + |Y| = n(G)$ and X and Y are independent sets, we must have $|X| = |Y| = n(G)/2$. If X contains an isolated vertex v of G, then $Y \cup \{v\}$ is again an independent set of size $n(G)/2 + 1$, a contradiction. So X and similarly Y (and hence G) do not contain isolated vertices. But then $\beta'(G) = \alpha(G) = n(G)/2$. Since $\alpha'(G) + \beta'(G) = n(G)$, we have $\alpha'(G) = n(G)/2$, i.e., a maximum matching of G saturates every vertex of X and of Y, and hence is a perfect matching.

5. The blocks of a graph are those of its components; so it is sufficient to prove the equation for a connected graph G, i.e., for the case $k = 1$. We know that a vertex $v \in V(G)$ belongs to two (or more) blocks of G if and only if v is a cut-vertex of G. Therefore, we can write $\sum_{v \in V(G)} [b(v) - 1] = \sum_{i=1}^{r} [b(v_i) - 1] = [\sum_{i=1}^{r} b(v_i)] - r$, where $v_1, \ldots, v_r$ are all the cut-vertices of G. Consider the block-cutpoint graph $B(G)$ of G. It is a bipartite graph. Counting its edges using endpoints in the partite set $\{v_1, \ldots, v_r\}$ gives $e(B(G)) = \sum_{i=1}^{r} b(v_i)$. Since G is connected, $B(G)$ is a tree and hence $e(B(G)) = n(B(G)) - 1 = b(G) + r - 1$. Thus $b(G) - 1 = e(B(G)) - r = [\sum_{i=1}^{r} b(v_i)] - r$.

6. **(a)** First consider $\delta(G) = n - 1$. Then G is K_n and has connectivity $n - 1$. So assume that $\delta(G) = n - 2$. Let S be a vertex-cut of G. Suppose that S has $\leqslant n - 3$ vertices. $G \setminus S$ is disconnected and contains at least 3 vertices; so we can find distinct $u, v, w \in V(G) \setminus S$ such that v and w are not neighbors of u in $G \setminus S$ and hence in G as well. G being simple, u is also not a neighbor of itself. Thus $\delta(G) \leqslant |N(u)| \leqslant n - 3$, a contradiction. Thus $|S| \geqslant n - 2$, i.e., every vertex cut of G has at least $n - 2$ vertices, i.e., $\kappa(G) \geqslant n - 2$. On the other hand, $\kappa(G) \leqslant \delta(G) = n - 2$. So $\kappa(G) = n - 2$.

(b) Let $[S, \overline{S}]$ be an edge-cut of G with $|S| = k$ $(1 \leqslant k \leqslant n - 1)$. Then $|[S, \overline{S}]| \geqslant k\delta(G) - k(k-1)/2 = \delta(G) + (k-1)(\delta(G) - k/2) \geqslant \delta(G) + (k-1)(\lfloor n/2 \rfloor - (n-1)/2) \geqslant \delta(G)$. That is, every edge-cut of G contains at least $\delta(G)$ edges, implying that $\kappa'(G) \geqslant \delta(G)$. Since $\kappa'(G) \leqslant \delta(G)$ always, we have $\kappa'(G) = \delta(G)$.

7. The following figure shows a planar embedding of G for the case $n = 8$. It is clear that this construction can be done for all $n \geqslant 3$. More explicitly, start with the three cycle for $n = 3$. So let $n \geqslant 4$ and inductively assume that the graph $G \setminus v_n$ contains a face whose boundary is the 3-cycle containing the vertices $v_{n-3}, v_{n-2}, v_{n-1}$. Put v_n in the interior of this face and join v_n to $v_{n-3}, v_{n-2}, v_{n-1}$. This again creates a face whose boundary is the 3-cycle containing the vertices v_{n-2}, v_{n-1}, v_n.



We have:

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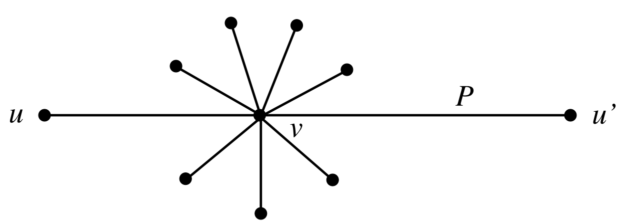
\begin{aligned}
e(G) &= |\{v_iv_{i+1} \mid 1 \leqslant i \leqslant n-1\}| + |\{v_iv_{i+2} \mid 1 \leqslant i \leqslant n-2\}| + |\{v_iv_{i+3} \mid 1 \leqslant i \leqslant n-3\}| \\
&= (n-1)+(n-2)+(n-3) \\
&= 3n-6.
\end{aligned}
$$

Thus G is a maximal planar graph.

8. Since loops and multiple edges do not affect existence (and diameters) of spanning trees in a graph, we may assume that G is simple. Let T_2 and T_l be spanning trees of G of diameters 2 and l respectively. We are supposed to construct a spanning tree T_k of G of diameter k with $2 \leqslant k \leqslant l$.

Let $n := n(G)$. Since T_2 has at least three vertices, $n \geqslant 3$. T_2 is the star $K_{1,n-1}$. Let v be the center of T_2 (i.e., the only non-leaf vertex in T_2). Since T_2 is a subgraph of G, v is adjacent to any vertex $u \in V(G)\setminus\{v\}$. T_l contains a path of length l. Take a sub-path P of length k in this path. Let u, u' be the endpoints of P. Consider the following three cases:

Case 1 : v is an internal vertex of P.
Take $E(T_k) = E(P) \cup \{vw \mid w \in V(G) \setminus V(P)\}$.



Case 2 : v is one endpoint of P.

Because of symmetry we can assume that $v = u$. Let w' be *the* neighbor of u' on P. Delete the edge $w'u'$ from P and add the edge $u'v = u'u$ to P. This gives us a u', w'-path of length k in which v is an internal vertex. Now use Case 1.

Case 3 : v is not on P.

Let w be *the* neighbor of u on P. Delete the vertex u (and the edge uw) from P and add the vertex v and the edge vw to P. This gives us a v, u'-path of length k. Now use Case 2.