CS30053 Foundations of Computing, Autumn 2005

Class test 2 : Solutions

Roll No: ____________ **Name:** ______________________________

Answer all questions in the respective spaces provided.
Use extra sheets for rough work. Any such extra sheet will not be corrected.

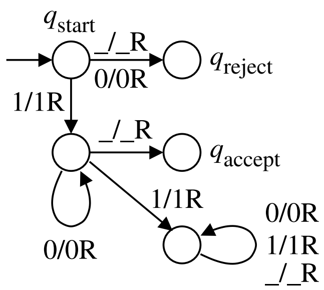
1. (a) Let M be a Turing machine with $\mathcal{L}(M) = L$ and with exactly one accept state and exactly one reject state. Construct a Turing machine N by swapping the accept and reject states of M. Prove or disprove: $\mathcal{L}(N) = \overline{L}$. **(5)**

Solution The assertion is false in general. Consider the pairwise disjoint languages:

$$\begin{aligned} L_a &= \{\alpha \in \Sigma^* \mid M \text{ accepts } \alpha\} \\ L_r &= \{\alpha \in \Sigma^* \mid M \text{ rejects } \alpha \text{ after halting}\} \\ L_l &= \{\alpha \in \Sigma^* \mid M \text{ loops on } \alpha\} \end{aligned}$$

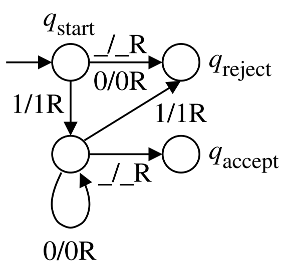
Then $\mathcal{L}(M) = L_a$, $\mathcal{L}(N) = L_r$, and $L_a \cup L_r \cup L_l = \Sigma^*$. Unless $L_l = \emptyset$, we cannot say $\mathcal{L}(N) = \overline{L}$.

(b) Prove or disprove: The language recognized by the Turing machine shown below is Turing-decidable. **(5)**



Solution The language, call it L, is Turing-decidable. In fact, L is the language of the regular expression 10^*. We know that regular languages are Turing-decidable.

[The given TM is not a decider for L, since it does not halt, for example, on the input 101. A decider for L is given below:



For solving this exercise, designing such a decider is not necessary.]

2. (a) Prove that Turing-recognizable languages are closed under union. **(5)**

Solution Let $L_1 = \mathcal{L}(M_1)$ and $L_2 = \mathcal{L}(M_2)$ be two Turing-recognizable languages, where M_1 and M_2 are TMs. Let us design a TM M with $\mathcal{L}(M) = L_1 \cup L_2$. M runs M_1 and M_2 in "parallel". M may be designed as a two-tape machine. To start with M copies the input α to the second tape and positions both the heads at the beginning of the respective copies of the input. M simulates the behavior of M_1 on the first tape and that of L_2 on the second tape *simultaneously*. M accepts if and only if either the simulation of M_1 or that of M_2 accepts. If both M_1 and M_2 reject α after halting, then M also rejects. If one of M_1 and M_2 rejects after halting and the other by not halting, or if both M_1 and M_2 reject by not halting, then M rejects by not halting.

(b) Let $L_1, L_2, \ldots, L_n$ be pairwise disjoint Turing-recognizable languages over the same alphabet Σ. Suppose that $\bigcup_{i=1}^{n} L_i = \Sigma^*$. Prove that each L_i is Turing-decidable. **(5)**

Solution By Part (a) Turing-recognizable languages are closed under finite union. It follows that $\bigcup_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} L_j$ is Turing-recognizable, i.e., $\bar{L}_i$ is Turing-recognizable. Since L_i is also Turing-recognizable, it follows that L_i is Turing-decidable.