CS30053 Foundations of Computing, Spring 2004

Mid-Semester Examination

Total points: 30 | September 21, 2004 | Total time: 2 hours

Answer any five questions. If you answer more, you must clearly indicate which five questions I would evaluate. If you fail to mention that, I will consider the five questions in which you score least points.
This exam is open-notes. You may bring any amount of hand-written material.

1. Let p, q, r be propositions. Show that: **(2×3)**

(a) $(\neg p \wedge q) \vee (q \rightarrow p)$ is a tautology.

(b) $(p \rightarrow q) \wedge (p \wedge \neg q)$ is a contradiction.

(c) $(p \wedge q) \vee r$ is not equivalent to $p \wedge (q \vee r)$.

2. Let a, b, c be positive integers. Prove that:

(a) Given $x \in \mathbb{R}$, there exist an integer q and a real number r such that $x = qc + r$ and $0 \leqslant r < c$. **(3)**

(b) $\lfloor \lfloor x/a \rfloor / b \rfloor = \lfloor x/(ab) \rfloor$ for all $x \in \mathbb{R}$. **(3)**

3. Let $A := \{1, 2, 3, \ldots, n\}$ and $f : A \rightarrow A$ a bijective function. Prove that the following three conditions are equivalent: **(6)**

(1) f is *not* the identity function ι_A.

(2) There exists $i \in A$ with $f(i) > i$.

(3) There exists $j \in A$ with $f(j) < j$.

(You may prove the implications (1) $\Longleftrightarrow$ (2) and (1) $\Longleftrightarrow$ (3).)

4. A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is recursively defined as:

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 2, \\ f(n) &= 5f(n-1) - 6f(n-2) + 1 \text{ for } n \geqslant 2. \end{aligned}$$

(a) Prove by induction on n that $2f(n-1) \leqslant f(n) \leqslant 3f(n-1) - 1$ for all $n \geqslant 1$. **(4)**

(b) Conclude that $f(n)$ is $O(3^n)$ and $\Omega(2^n)$. **(2)**

5. Let F_n, $n \in \mathbb{N}$, be the sequence of Fibonacci numbers. Prove that: **(2×3)**

(a) $F_{2n+1} = 1 + F_2 + F_4 + F_6 + \cdots + F_{2n}$ for all $n \in \mathbb{N}$.

(b) $F_{2n+2} = F_1 + F_3 + F_5 + \cdots + F_{2n+1}$ for all $n \in \mathbb{N}$.

(c) $\gcd(F_n, F_{n+1}) = 1$ for all $n \in \mathbb{N}$.

6. Let $a_1, a_2, \ldots, a_n$ be positive integers with $\sum_{i=1}^{n} a_i < 2^n - 1$. Prove that there exist distinct disjoint non-empty subsets A, B of $\{a_1, a_2, \ldots, a_n\}$ with the property that $\sum_{a \in A} a = \sum_{b \in B} b$. **(6)**

7. Let $k \in \mathbb{N}$. Show that the total number of solutions with non-negative integer values of the variables of:

(a) $x_1 + 2x_2 = 2k$ is $k + 1$. **(2)**

(b) $x_1 + 2x_2 + 4x_3 = 4k$ is $(k + 1)^2$. **(4)**