# CS30053 Foundations of Computing, Spring 2004

## Mid-Semester Examination : Solutions

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**1.** **(a)** $q \to p \equiv \neg q \vee p \equiv \neg(\neg p \wedge q)$. So the given expression is equivalent to $t \vee \neg t$, where $t := \neg p \wedge q$.

**(b)** As in Part (a) the given expression is equivalent to $t \wedge \neg t$, where $t := p \wedge \neg q$.

**(c)** For $p = \mathrm{F}$ and $r = \mathrm{T}$, $(p \wedge q) \vee r$ evaluates to T, whereas $p \wedge (q \vee r)$ evaluates to F.

**2.** **(a)** Let $q := \lfloor x/c \rfloor$. We can write $x/c = q + \epsilon$ with $0 \leqslant \epsilon < 1$. But then $x = qc + r$, where $0 \leqslant r := c\epsilon < c$.

**(b)** Take $c := ab$ and write $x = qab + r$, where $q = \lfloor x/(ab) \rfloor$ and $0 \leqslant r < ab$. But then $x/a = qb + (r/a)$ and so $\lfloor x/a \rfloor = qb + \lfloor r/a \rfloor$, i.e., $\lfloor \lfloor x/a \rfloor / b \rfloor = q + \lfloor \lfloor r/a \rfloor / b \rfloor$. Now $\lfloor r/a \rfloor \leqslant r/a < ab/a = b$ and $\lfloor r/a \rfloor \geqslant 0$. Therefore, $\lfloor \lfloor r/a \rfloor / b \rfloor = 0$.

**3.** $[\neg(1) \Rightarrow \neg(2)]$ Obvious.

$[\neg(2) \Rightarrow \neg(1)]$ Assume that $f(i) \leqslant i$ for all $i = 1, 2, \ldots, n$. Suppose that $f(i) < i$ for some $i$. Since $f$ is bijective, $f(1), f(2), \ldots, f(n)$ is a permutation of $1, 2, \ldots, n$. But then $1 + 2 + \cdots + n = \sum_{i=1}^{n} f(i) < \sum_{i=1}^{n} i$, which is absurd. So $f(i) = i$ for all $i$, i.e., $f = \iota_A$.

The implications $\neg(1) \iff \neg(3)$ can be proved similarly.

**4.** **(a)** The inequalities clearly hold for $n = 1$. So take $n \geqslant 2$ and assume that the inequalities hold for $n-1$, i.e., $2f(n-2) \leqslant f(n-1) \leqslant 3f(n-2) - 1$. It then follows that $2f(n-1) + 2 \leqslant 6f(n-2) \leqslant 3f(n-1)$. Now $f(n) = 5f(n-1) - 6f(n-2) + 1$, i.e., $f(n) \geqslant 5f(n-1) - 3f(n-1) + 1 = 2f(n-1) + 1 \geqslant 2f(n-1)$. Also $f(n) \leqslant 5f(n-1) - (2f(n-1) + 2) + 1 = 3f(n-1) - 1$.

**(b)** For all $n \in \mathbb{N}$ we have $f(n) \leqslant 3f(n-1) - 1 \leqslant 3f(n-1) \leqslant 3^2 f(n-2) \leqslant 3^3 f(n-3) \leqslant \cdots \leqslant 3^n f(0) = 3^n$. Similarly, $f(n) \geqslant 2f(n-1) \geqslant 2^2 f(n-2) \geqslant 2^3 f(n-3) \geqslant \cdots \geqslant 2^n f(0) = 2^n$.

**5.** Let me prove the assertions by induction on $n$. All the assertions are clearly true for $n = 0$. So let me prove only the inductive steps. The expressions in square brackets follow from the induction hypotheses.

**(a)** $F_{2(n+1)+1} = F_{2n+3} = F_{2n+1} + F_{2n+2} = [1 + F_2 + F_4 + \cdots + F_{2n}] + F_{2n+2} = 1 + F_2 + F_4 + \cdots + F_{2(n+1)}$.

**(b)** $F_{2(n+1)+2} = F_{2n+4} = F_{2n+2} + F_{2n+3} = [F_1 + F_3 + \cdots + F_{2n+1}] + F_{2n+3} = F_1 + F_3 + \cdots + F_{2(n+1)+1}$.

**(c)** $\gcd(F_{n+1}, F_{n+2}) = \gcd(F_{n+1}, F_{n+2} - F_{n+1}) = \gcd(F_{n+1}, F_n) = \gcd(F_n, F_{n+1}) = [1]$.

**6.** Let $S := \{a_1, a_2, \ldots, a_n\}$. Since $S$ has $2^n - 1$ non-empty subsets and the sum $\sum_{x \in A} x$ for a non-empty subset $A$ of $S$ can assume $\leqslant 2^n - 2$ values (between 1 and $2^n - 2$), by the pigeon-hole principle $\sum_{a \in A'} a = \sum_{b \in B'} b$ for two distinct non-empty subsets $A'$ and $B'$ of $S$. Now throw away the common elements from $A'$ and $B'$, i.e., take $A := A' \setminus B' = A' \setminus (A' \cap B')$ and $B := B' \setminus A' = B' \setminus (A' \cap B')$. Then $A$ and $B$ are non-empty too and we continue to have $\sum_{a \in A} a = \sum_{b \in B} b$.

**7.** **(a)** For each value of $x_2 \in \{0, 1, \ldots, k\}$ we get a unique non-negative solution for $x_1$. For $x_2 > k$ the value of $x_1$ becomes negative (and so unacceptable).

**(b)** If $x_3 > l$, we cannot have non-negative solutions for both $x_1$ and $x_2$. For $x_3 = l \in \{0, 1, \ldots, k\}$ we have $x_1 + x_2 = 2(2k - 2l)$, which has $2k - 2l + 1$ solutions for $(x_1, x_2)$ by Part (a). Thus the total number of solutions of the original equation is $\sum_{l=0}^{k} (2k - 2l + 1) = (2k+1) + (2k-1) + \cdots + 5 + 3 + 1 = (k+1)^2$.

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