CS30053 Foundations of Computing, Spring 2004

Practice exercises : Set 4

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1. Define a recurrence relation and the requisite number of initial conditions for each of the following:

(a) The number of binary strings of length n, that do not contain three consecutive 0's.

* **(b)** The number of binary strings of length n, that do not contain k consecutive 0's, where $k \in \mathbb{Z}^+$ is a constant. (**Hint:** Look at the first occurrence of 1.)

2. How many initial conditions are needed for completely specifying the sequences defined by the following recurrence relations? Also explicitly mention for which values of n one should specify these initial conditions. Assume that each sequence a_n in this exercise starts from $n = 0$.

(a) $a_n = a_{n-2}$.

(b) $a_n = a_2$.

(c) $a_n = 2a_{n-2}^2 + 3a_{n-3}^3 + 4^{n-4}$.

(d) $a_n = (n-1)a_{n-1} + (n-2)a_{n-2} + \cdots + 2a_2 + a_1$.

(e) $a_n = na_{n-1} + (n-1)a_{n-2} + \cdots + 3a_2 + 2a_1 + a_0$.

(f) $a_n = a_{\lfloor n/2 \rfloor} a_{\lceil n/2 \rceil} + 1$.

3. Solve the following recurrence relations:

(a) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-1} + 12a_{n-2}$ for $n \geqslant 2$.

(b) $a_0 = 2$, $a_1 = 3$, $6a_n = a_{n-1} + 12a_{n-2}$ for $n \geqslant 2$.

(c) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_n = a_{n-1} + 4a_{n-2} - 4a_{n-2}$ for $n \geqslant 3$.

(d) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_n = 2a_{n-2} - a_{n-4}$ for $n \geqslant 4$.

(e) $a_0 = 2$, $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_n = 3a_{n-2} - 2a_{n-4}$ for $n \geqslant 4$.

(f) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)$ for $n \geqslant 1$.

(g) $a_0 = 2$, $a_n = 5a_{n-1} + (n^2 + n + 1)2^n$ for $n \geqslant 1$.

(h) $a_0 = 2$, $a_1 = 3$, $a_n = 2(a_{n-1} + a_{n-2} + 2^n)$ for $n \geqslant 2$.

(i) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + 2^n)$ for $n \geqslant 2$.

(j) $a_0 = 2$, $a_1 = 3$, $a_n = 4(a_{n-1} - a_{n-2} + n2^{2n-1})$ for $n \geqslant 2$.

4. Reduce the following recurrence relations to standard forms and solve:

(a) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n+2} - a_{n+1} - n$ for all $n \geqslant 0$.

(b) $a_0 = 2$, $a_1 = 3$, $4a_{n+1} + 8a_n - 5a_{n-1} = 2^n$ for all $n \geqslant 1$.

(c) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-1} + 12(a_{n-2} + 2^{n-2} + 1)$ for all $n \geqslant 2$.

(d) $a_0 = 2$, $a_n^3 = a_{n-1}(3a_n^2 - 3a_n a_{n-1} + a_{n-1}^2) + n^3$ for all $n \geqslant 1$.

* **(e)** $a_0 = 2$, $a_1 = 3$, $2a_n a_{n-2} - 2a_{n-1}^2 - 3a_{n-1}a_{n-2} = 0$ for all $n \geqslant 2$.

(f) $a_0 = 2$, $a_1 = 3$, $2^{a_n} = 4^n \times 16^{a_{n-2}}$ for all $n \geqslant 2$.

5. Let b_n be a sequence satisfying the recurrence

$$b_n = d_1 b_{n-1} + d_2 b_{n-2} + \cdots + d_k b_{n-k} + F(n) \text{ for } n \geqslant k,$$

and c_n another sequence satisfying the recurrence

$$c_n = d_1 c_{n-1} + d_2 c_{n-2} + \cdots + d_k c_{n-k} + G(n) \text{ for } n \geqslant k,$$

where $k \in \mathbb{Z}^+$ and $d_1, d_2, \ldots, d_k \in \mathbb{R}$ are constants. Consider the sequence $a_n := b_n + c_n$.

(a) Show that the sequence a_n satisfies the recurrence

$$a_n = d_1 a_{n-1} + d_2 a_{n-2} + \cdots + d_k a_{n-k} + F(n) + G(n) \text{ for } n \geqslant k. \tag{1}$$

(b) Let $F(n) = p(n)s^n$ and $G(n) = q(n)t^n$, where $p(n)$ and $q(n)$ are polynomials with real coefficients and where s and t are distinct non-zero real constants. Give a particular solution of (1).

Solve the following recurrences:

(c) $a_0 = 2$, $a_n = 2a_{n-1} + 2^n + n^2$ for all $n \geqslant 1$.

(d) $a_0 = 2$, $a_1 = 3$, $a_n = 2a_{n-1} - a_{n-2} + 2^n + n^2$ for all $n \geqslant 2$.

(e) $a_0 = 2$, $a_1 = 3$, $a_n = a_{n-2} + 2^n + n3^n + n^2 4^n$ for all $n \geqslant 2$.

6. Find big-O estimates for the following positive-integer-valued increasing functions $f(n)$.

(a) $f(n) = 125 f(n/4) + 2n^3$ whenever $n = 4^k$ for $k \in \mathbb{Z}^+$.

(b) $f(n) = 125 f(n/5) + 2n^3$ whenever $n = 5^k$ for $k \in \mathbb{Z}^+$.

(c) $f(n) = 125 f(n/6) + 2n^3$ whenever $n = 6^k$ for $k \in \mathbb{Z}^+$.

7. Let $f(n)$ be an increasing positive-real-valued function of a non-negative integer variable n. Give a big-O estimate of $f(n)$ for each of the following cases:

(a) $f(n) = 2f(\sqrt{n}) + 1$ whenever n is a perfect square bigger than 1.

(b) $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square bigger than 1.

(c) $f(n) = 2f(\sqrt{n}) + \log^2 n$ whenever n is a perfect square bigger than 1.

(d) $f(n) = af(\sqrt[b]{n}) + c(\log n)^d$ whenever n is a perfect b-th power bigger than 1. Here $a, b \in \mathbb{N}$, $a \geqslant 1$, $b \geqslant 2$, $c, d \in \mathbb{R}$, $c > 0$ and $d \geqslant 0$.

8. Use generating functions to prove the following identities:

(a) $\displaystyle\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$. **(b)** $\displaystyle\binom{m}{r} = \sum_{k=0}^{2r} (-1)^{r+k} \binom{m}{k}\binom{m}{2r-k}$.

(c) $\displaystyle\binom{m}{r} = \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k \binom{m}{k}\binom{m+r-2k-1}{r-2k}$.

9. Use generating functions to solve the following recurrence relations:

(a) $a_0 = 2$, $a_n = 5a_{n-1} + 4n + 3$ for $n \geqslant 1$.

(b) $a_0 = 2$, $a_1 = 3$, $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geqslant 2$.

(c) $a_0 = 2$, $a_1 = 3$, $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ for $n \geqslant 2$.

(d) $a_0 = 2$, $a_1 = 3$, $a_n = 3a_{n-1} - 2a_{n-2} + 2^n$ for $n \geqslant 2$.

(e) $a_0 = 2$, $a_1 = 3$, $a_n = 3a_{n-1} - 2a_{n-2} + 2^n + 1$ for $n \geqslant 2$.

(f) $a_0 = 2$, $a_1 = 3$, $a_n = 4a_{n-1} - 4a_{n-2} + 2^n + 1$ for $n \geqslant 2$.

10. Let $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, $b(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ and $c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ be power series. Show that:

(a) $a(x)(b(x) + c(x)) = a(x)b(x) + a(x)c(x)$.

(b) $a(x)$ is invertible (i.e., there exists a power series $d(x)$ with $a(x)d(x) = 1$) if and only if $a_0 \neq 0$.