CS30053 Foundations of Computing, Spring 2004

Practice exercises : Set 3

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1. Count the numbers of bit-strings of length ten that:

(a) start with 01 and end with 10.
(b) start with 01 and do not end with 10.
(c) neither start with 01 nor end with 10.
(d) contain neither 01 nor 10 as a substring.
(e) contain 01 but not 10 as a substring.
(f) contain both 01 and 10 as substrings.
(g) contain equal number of 0's and 1's.
(h) contain more 0's than 1's.

2. Count the numbers of positive integers less than or equal to 1000 that are:

(a) divisible by 5 or 7 (or both).
(b) divisible by both 5 and 7.
(c) divisible by neither 5 nor 7.
(d) divisible by 5 but not by 7.
(e) divisible by 6 or 8 (or both).
(f) divisible by both 6 and 8.
(g) divisible by neither 6 nor 8.
(h) divisible by 6 but not by 8.

3. Prove that if five points are placed inside an equilateral triangle of side 1 cm, there exist two of these points, that are no more than $1/2$ cm apart.

4. Let n be an odd positive integer and π a permutation of $1, 2, \ldots, n$, i.e., a bijective function $A \to A$, where $A := \{1, 2, \ldots, n\}$. Prove that the product $\prod_{i=1}^{n} (i - \pi(i))$ is even. (**Hint:** Look at the $(n+1)/2$ images $\pi(1), \pi(3), \pi(5), \ldots, \pi(n)$.) Show that the result need not hold if n is even.

5. Prove that in any group of $n \geqslant 2$ persons, there exist two persons having equal number of acquaintances (among the given n persons). Assume that acquaintance is a symmetric relation, i.e., two persons are either both acquainted or both strangers to one another.

6. Let $A \subseteq \{1, 2, \ldots, 2n\}$ with $|A| = n + 1$. Prove that:

(a) There exist $x_1, y_1 \in A$ such that $x_1 - y_1 = 1$.
(b) There exist $x_2, y_2 \in A$ such that $x_2 - y_2 = n$.
* **(c)** There exist $x_3, y_3 \in A$ such that $\gcd(x_3, y_3) = 1$.

7. Let $a_1, a_2, \ldots, a_n$ be n integers (not necessarily all distinct). Prove that there exists a non-empty sub-collection $a_{i_1}, \ldots, a_{i_k}$ with $k \geqslant 1$ and $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that $n \mid (a_{i_1} + \cdots + a_{i_k})$.

** **8.** Let F_n, $n \in \mathbb{N}$, be the sequence of Fibonacci numbers and let $m \in \mathbb{Z}^+$. Prove that there exists $n \in \mathbb{Z}^+$ such that $m \mid F_n$. (**Hint:** Look at the remainder sequence r_n, $n \in \mathbb{N}$, where $r_n := F_n \operatorname{rem} m$.)

* **9.** Let $f(X)$ be a polynomial with integer coefficients such that $f(a) = f(b) = f(c) = 2$ for three distinct integers a, b, c. Prove that $f(n) \neq 1$ for all $n \in \mathbb{Z}$. (**Hint:** First show that $(m - n) \mid (f(m) - f(n))$ for any two integers m, n.)

10. Count the numbers of solutions of the following:

(a) $x_1 + x_2 + x_3 + x_4 = 56$ with non-negative integers x_1, x_2, x_3, x_4.
(b) $x_1 + x_2 + x_3 + x_4 = 56$ with positive integers x_1, x_2, x_3, x_4.
(c) $x_1 + x_2 + x_3 + x_4 = 56$ with integers $x_1 \geqslant 1$, $x_2 \geqslant 2$, $x_3 \geqslant 3$, $x_4 \geqslant 4$.
* **(d)** $x_1 + x_2 + x_3 + x_4 \leqslant 56$ with nonnegative integers x_1, x_2, x_3, x_4. (**Hint:** Introduce x_5.)
* **(e)** $x_1 + x_2 + x_3 + x_4 \leqslant 56$ with integers $x_1 \geqslant 1$, $x_2 \geqslant 2$, $x_3 \geqslant 3$, $x_4 \geqslant 4$.
* **(f)** $x_1 + x_2 + x_3 + x_4 \geqslant 56$ with integers $x_1 \leqslant 11$, $x_2 \leqslant 22$, $x_3 \leqslant 33$, $x_4 \leqslant 44$. (**Hint:** $y_i := 11i - x_i$.)

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