CS30053 Foundations of Computing, Autumn 2004

End-Semester Examination : Solutions

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1. (a) *False*. It is easy to see that $n = \Theta(n)$ and $-n = \Theta(n)$. But $0 = n + (-n)$ is not $\Theta(2n)$.

However, if we concentrate only on non-negative real-valued functions, the given assertion is true. By hypothesis there exist positive real constants c_1, c_2, d_1, d_2 and natural numbers N_1, N_2 such that:

$$c_1 g_1(n) \leqslant f_1(n) \leqslant d_1 g_1(n) \quad \forall n \geqslant N_1$$
$$c_2 g_2(n) \leqslant f_2(n) \leqslant d_2 g_2(n) \quad \forall n \geqslant N_2$$

Let $c := \min(c_1, c_2)$, $d := \max(d_1, d_2)$ and $N := \max(N_1, N_2)$. Here c and d are positive real constants and N is a natural number. Moreover,

$$c(g_1(n) + g_2(n)) \leqslant (f_1(n) + f_2(n)) \leqslant d(g_1(n) + g_2(n)) \quad \forall n \geqslant N\,.$$

(b) *True*. A string of length n containing 01 as a substring can be written as $\alpha 01 \beta$ for any string $\alpha, \beta \in \{0,1\}^*$ with $|\alpha| + |\beta| = n - 2$. For a given length l of α in $\{0, 1, \ldots, n-2\}$ the length of β becomes fixed ($n - 2 - l$), and total choices for α and β are $2^l \times 2^{n-2-l} = 2^{n-2}$. Since there are $n - 1$ choices for l, we get a total of $(n-1)2^{n-2}$ strings of length n with 01 as a substring. However, for $n \geqslant 4$, some strings are counted more than once. For example, any string of the form 0101γ is counted (at least) twice, once as $(01)01(\gamma)$ and once as $(\lambda)01(01\gamma)$. In view of this, the exact number a_n is strictly less than the above count $(n-1)2^{n-2}$ for $n \geqslant 4$.

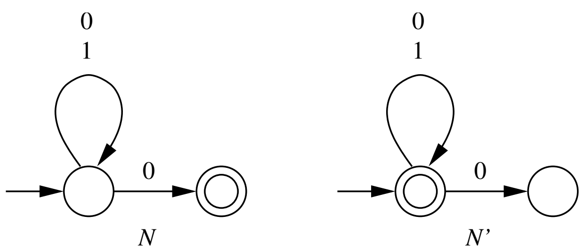
The exact recurrence for a_n can be derived based on the *first* occurrence of 01. Thus a decomposition of the form $\alpha 01 \beta$ with α *not* containing 01 is unique. This gives

$$a_n = (2^0 - a_0)2^{n-2} + (2^1 - a_1)2^{n-3} + (2^2 - a_2)2^{n-4} + \cdots + (2^{n-3} - a_{n-3})2^1 + (2^{n-2} - a_{n-2})2^0$$

for all $n \geqslant 2$.

(c) *False*. The string 0011 contains equal number of 0's and 1's, but cannot be generated by the regular expression $((01) \cup (10))^*$.

(d) *False*. For the example below, the string 0 belongs to both $\mathcal{L}(N)$ and $\mathcal{L}(N')$.



(e) *False*. Assume that the given language, call it L, is regular and let n be a pumping-lemma constant for L. Then $\alpha := a^n c b^n \in L$ and so by the pumping lemma, we get a decomposition $\alpha = \beta_1 \beta_2 \beta_3$ with β_2 non-empty and consisting only of a's. But then $\beta_1 \beta_3$ contains more b's than a's and still is in L, a contradiction.

(f) *True*. The given language, call it L, is equal to Σ^* (where $\Sigma = \{a, b\}$) which is clearly regular. For the proof of the fact that any string $\gamma \in \Sigma^*$ can be decomposed as $\gamma = \alpha\beta$ with the number of a's in α equal to the number of b's in β, we proceed by induction on $|\gamma|$. If $|\gamma| = 0$, the decomposition $\gamma = \lambda\lambda$ suffices. So assume that $|\gamma| \geqslant 1$ and that all strings of length $|\gamma| - 1$ belong to L. Consider the following two cases:

Case 1: $\gamma = b\gamma'$. By induction $\gamma' = \alpha'\beta'$ is a decomposition of γ' with the given property. Take $\alpha := b\alpha'$ and $\beta := \beta'$.

Case 2: $\gamma = a\gamma'$. Again let $\gamma' = \alpha'\beta'$ be a suitable decomposition of γ'. If $\alpha' = \lambda$, then β' is of the form a^k for some $k \in \mathbb{N}$. But then $\gamma = a^{k+1}$ and we can take $\alpha := \lambda$ and $\beta := a^{k+1}$. If $\alpha' = \alpha'' a$, take $\alpha := a\alpha''$ and $\beta := a\beta'$. Finally, if $\alpha' = \alpha'' b$, take $\alpha := a\alpha''$ and $\beta := b\beta'$.

2. **(a)** Clearly, $(x, y) = 1\cdot(x, y)$. So ρ is reflexive. If $(x', y') = c(x, y)$ for some $c \neq 0$, then $(x, y) = \frac{1}{c}(x', y')$ with $\frac{1}{c} \neq 0$; so ρ is symmetric. Finally, if $(x', y') = c(x, y)$ and $(x'', y'') = c'(x', y')$ with nonzero c, c', then $(x'', y'') = c'c(x, y)$ with $c'c \neq 0$, i.e., ρ is transitive.

(b) $f(0) = f(0 + 0) = f(0) + f(0)$, so that $f(0) = 0$. Also $f(1) = f(1 \times 1) = f(1)f(1)$, i.e., $f(1) = 0, 1$. Since f is injective and $f(0) = 0$, we have $f(1) = 1$. By induction on n we can then show that $f(n) = f(1) + f(n - 1) = 1 + (n - 1) = n$ for all $n \in \mathbb{N}$. Moreover, since $0 = f(0) = f(n + (-n)) = f(n) + f(-n)$, it follows that $f(-n) = -n$ for all $n \in \mathbb{N}$. Finally, let $n/m \in \mathbb{Q}$ with $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. We have $n = f(n) = f((n/m) \times m) = mf(n/m)$ and so $f(n/m) = n/m$.

3. **(a)** Let α be a string over $\{0, 1, 2\}$ of length n not containing two consecutive 0's. Let $n \geqslant 2$. If the first symbol of α is 1 or 2, the remaining part of α may be any string of length $n - 1$ not containing two consecutive 0's. However, if the first symbol of α is 0, the second symbol must be either 1 or 2, and the remaining $n - 2$ symbols can form any string not containing two consecutive 0's. It then follows that

$$a_n = 2a_{n-1} + 2a_{n-2} \quad \text{for } n \geqslant 2\,.$$

The boundary conditions are:

$$\begin{aligned} a_0 &= 1 \quad \text{(The empty string does not contain two consecutive 0's.)} \\ a_1 &= 3 \quad \text{(Each string of length 1 does not contain two consecutive 0's.)} \end{aligned}$$

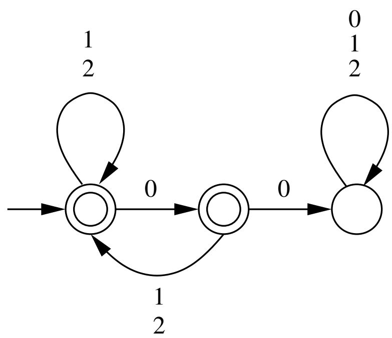
(b) The characteristic equation $x^2 - 2x - 2 = 0$ has roots $1 \pm \sqrt{3}$, i.e., $a_n = A(1 + \sqrt{3})^n + B(1 - \sqrt{3})^n$ for some A, B. Plugging in the boundary conditions gives $A = \frac{2+\sqrt{3}}{2\sqrt{3}}$ and $B = -\frac{2-\sqrt{3}}{2\sqrt{3}}$. Therefore,

$$\begin{aligned} a_n &= \frac{1}{2\sqrt{3}}\left[(2 + \sqrt{3})(1 + \sqrt{3})^n - (2 - \sqrt{3})(1 - \sqrt{3})^n\right] \\ &= \frac{1}{4\sqrt{3}}\left[(1 + \sqrt{3})^{n+2} - (1 - \sqrt{3})^{n+2}\right] \quad \text{for all } n \in \mathbb{N}\,. \end{aligned}$$

4. **(a)** The regular grammar $G = (V, \Sigma, S, P)$ defines the given language, where $V = \{S, T, 0, 1, 2\}$, $\Sigma = \{0, 1, 2\}$, and P consists of the following productions:

$$\begin{aligned} S &\to \lambda \mid 0 \mid 0T \mid 1S \mid 2S\,, \\ T &\to 1S \mid 2S\,. \end{aligned}$$

(b)



5. **(a)** $L := \mathcal{L}(G) = \{a^i b^j \mid i, j \in \mathbb{N}, 0 \leqslant j \leqslant 2i\}$. Well, I think this is perfect English!

(b) Suppose that L is regular and n a pumping lemma constant for L. Consider $\alpha := a^n b^{2n} \in L$. By the pumping lemma we have $\alpha = \beta_1\beta_2\beta_3$ with β_2 non-empty and consisting of a's only. Moreover, L contains $\beta_1\beta_3 = a^m b^{2n}$ with $m < n$, a contradiction.