CS30053 Foundations of Computing, Spring 2004

Class test 1 : Solutions

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1. **(a)** *False*. For $p = \mathrm{F}$ and $q = \mathrm{T}$, the subexpression $p \rightarrow q \equiv \neg p \vee q$ evaluates to T and so $(p \rightarrow q) \wedge \neg p$ evaluates to T, whereas $\neg q$ is F.

(b) *True*. We have $(p \rightarrow q) \wedge \neg q \equiv (\neg p \vee q) \wedge \neg q \equiv (\neg p \wedge \neg q) \vee (q \wedge \neg q) \equiv (\neg p \wedge \neg q) \vee \mathrm{F} \equiv \neg p \wedge \neg q \equiv \neg(p \vee q)$.

(c) *True*. Given $x \in \mathbb{R}$ take $m := \lfloor x \rfloor - 1$ and $n := \lceil x \rceil + 1$.

(d) *False*. For $x \in \mathbb{R}$, $x \leqslant 0$, we do not have a non-negative integer $m < x$.

(e) *False*. Take $x = 2$. Then $\lceil \lfloor x/2 \rfloor /3 \rceil = \lceil \lfloor 2/2 \rfloor /3 \rceil = \lceil 1/3 \rceil = 1$, whereas $\lfloor \lceil x/2 \rceil /3 \rfloor = \lfloor \lceil 2/2 \rceil /3 \rfloor = \lfloor 1/3 \rfloor = 0$.

(f) *True*. Since $f(n) = n^2 + 2n + 3 = (n+1)^2 + 2$, $f(m) = f(n)$ implies $(m+1)^2 = (n+1)^2$. But $m+1$ and $n+1$ are both positive and so $m+1 = n+1$, i.e., $m = n$.

(g) *False*. As in Part (f) $g(m) = g(n)$ implies $(m+1)^2 = (n+1)^2$, but now we may have $m+1 = -(n+1)$, i.e., $m = -n-2 \neq n$ (unless $n = -1$). As a specific example, we have $g(0) = g(-2) = 3$.

2. Let $A = \{a_1, a_2, \ldots, a_n\}$.

(a) Consider an $i \in \{1, 2, \ldots, n\}$. There are two possibilities for (a_i, a_i)—it is either in the relation or not. Also for each i, j with $1 \leqslant i < j \leqslant n$ there are three possibilities that respect anti-symmetry: 1) neither (a_i, a_j) nor (a_j, a_i) belongs to the relation, 2) only (a_i, a_j) (but not (a_j, a_i)) belongs to the relation, and 3) only (a_j, a_i) (but not (a_i, a_j)) belongs to the relation.

(b) Let R be a relation on A, that is both symmetric and anti-symmetric. Let $i, j \in \{1, 2, \ldots, n\}$. Then $a_i \, R \, a_j$ implies $a_j \, R \, a_i$ by symmetry and so by anti-symmetry $i = j$. Thus R contains only pairs of the form (a_i, a_i) and for each such i there are two choices—either include (a_i, a_i) in R or not.

3. **(a)** Let $f(m) = f(n)$. If $m < n$, we have $f(m) < f(n)$, whereas if $m > n$, we have $f(m) > f(n)$. So we must have $m = n$.

(b) Consider $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ n & \text{if } n \geqslant 2. \end{cases}$

(c) If $f(n+1) - f(n) = 0$ for some n, then f is not injective. So assume that $f(n+1) - f(n) \in \{1, -1\}$ for all n. I now claim that $f(n+1) - f(n) = 1$ for all n, which proves that f is strictly monotonic increasing. For proving the claim assume that there exists an n, call it N, for which $f(N+1) - f(N) = -1$. Let $k := f(N)$, so that $f(N+1) = k - 1$. Now $f(N+2) = f(N+1) \pm 1$. Since f is given to be injective, this forces $f(N+2) = (k-1) - 1 = k - 2$. Similarly, we have $f(N+3) = k - 3$, $f(N+4) = k - 4$ and so on. In particular, $f(N+k-1) = 1$ and $f(N+k) = 0$. Since $f(N+k+1)$ must be non-negative and differ from $f(N+k)$ by (at most) 1, we must then have $f(N+k+1) = 1 = f(N+k-1)$, a contradiction to the injectivity of f.
