1. Design an unrestricted grammar for the following language: $\left\{ a^{n^2} \middle| n \geq 0 \right\}$

2. Design an unrestricted grammar for the following language.

$\{ ww \mid w \in \{a,b\}^* \}$

3. A shuffle of two strings α and β is a string γ of length $|\alpha| + |\beta|$, in which α and β are non-overlapping subsequences (not necessarily substrings). For example, all shuffles of ab and cd are abcd, cabd, cdab, acbd, acdb, and cadb. For two languages A and B, we define shuffle(A, B) as the language consisting of all shuffles of all $\alpha \in A$ and all $\beta \in B$. Prove that recursively enumerable languages are closed under the shuffle operation, that is, if A and B are r.e. languages, then so also is the language

$$\text{shuffle}(A, B) = \{\gamma \mid \gamma \text{ is a shuffle of some } \alpha \in A \text{ and } \beta \in B\}.$$

Is shuffle(A, B) recursive if A and B are recursive? Justify.

4. Prove that the problem whether a Turing machine M on a given input x reenters its start state is undecidable.

5. Consider the language

AL_{2022} = { M | M is a Turing machine which accepts at least 2022 input strings }.

(a) Prove that AL_{2022} is recursively enumerable.
(b) Prove that AL_{2022} is not recursive.

6. Let A be a language over Σ, and B a language over Λ. Suppose that $A \leqslant_m B$ under a reduction map $\Sigma^* \to \Lambda^*$ which is onto (surjective). Prove that $B \leqslant_m A$.