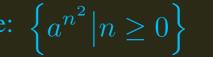
1. Design an unrestricted grammar for the following language:  $\left\{a^{n^2} | n \ge 0\right\}$ 



2. Design an unrestricted grammar for the following language.

 $\{ ww | w \in \{a,b\}^* \}$ 

3. A shuffle of two strings  $\alpha$  and  $\beta$  is a string  $\gamma$  of length  $|\alpha| + |\beta|$ , in which  $\alpha$  and  $\beta$  are non-overlapping subsequences (not necessarily substrings). For example, all shuffles of ab and cd are abcd, cabd, cdab, acbd, acdb, and cadb. For two languages A and B, we define shuffle(A, B) as the language consisting of all shuffles of all  $\alpha \in A$  and all  $\beta \in B$ . Prove that recursively enumerable languages are closed under the shuffle operation, that is, if A and B are r.e. languages, then so also is the language

shuffle(A, B) = { $\gamma \mid \gamma$  is a shuffle of some  $\alpha \in A$  and  $\beta \in B$ }.

Is shuffle(A, B) recursive if A and B are recursive? Justify.

4. Prove that the problem whether a Turing machine M on a given input x reenters its start state is undecidable.

## 5. Consider the language

 $AL_{2022} = \{ M \mid M \text{ is a Turing machine which accepts at least 2022 input strings } \}.$ 

(a) Prove that AL<sub>2022</sub> is recursively enumerable.(b) Prove that AL<sub>2022</sub> is not recursive.

6. Let A be a language over  $\Sigma$ , and B a language over  $\Lambda$ . Suppose that  $A \leq_m B$  under a reduction map  $\Sigma^* \to \Lambda^*$  which is onto (surjective). Prove that  $B \leq_m A$ .