1. One of the following sets is regular, the other is not. Justify.

A = { x$y | x, y in {a,b}* and #a(x) = #b(y) } ⊆ {a,b,$}*.
B = { xy | #a(x) = #b(y) } ⊆ {a,b}*.

Prove that this decomposition is unique

A is not regular. — Suppose A is regular. k a PLC for A

$$u=\epsilon,\quad v=a^k,\quad w=\$b^k.$$

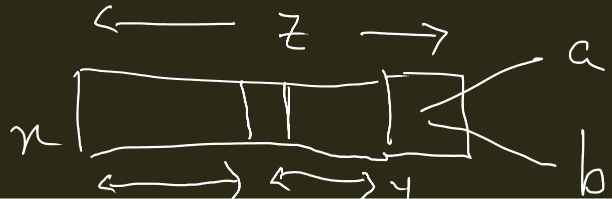
B is regular $\qquad u=\epsilon,\ v=a^k,\ w=b^k$

$$B \overset{\subseteq}{=} \{a,b\}^* \qquad a^{k-l}b^k = (a^{k-l}b^l)\,b^{k-l}$$

$$\supseteq \quad a^k \quad 0,1,2,\dots,k \qquad a^{k+l}b^k = a^k(a^l b^k)$$

Take $z \in \{a,b\}^*$. TST $z \in B$. $\qquad |z|=0 \quad z=\epsilon=\epsilon\epsilon$

Induction on $|z|$



✓ $\quad y=\epsilon \Rightarrow x=b^k,\ k\geq 0$

$y \neq \epsilon$

2. Let L be a regular language. Define dup(L) = { ww | w ∈ L }. Which of the following statements is true?

(a) dup(L) must be regular.
(b) dup(L) must be non-regular.
✓ (c) dup(L) may or may not be regular.

L finite

$L = \mathcal{L}(a^*)$

$dup(L) = \mathcal{L}((aa)^*)$

$L = \mathcal{L}((a+b)^*)$

PL proof

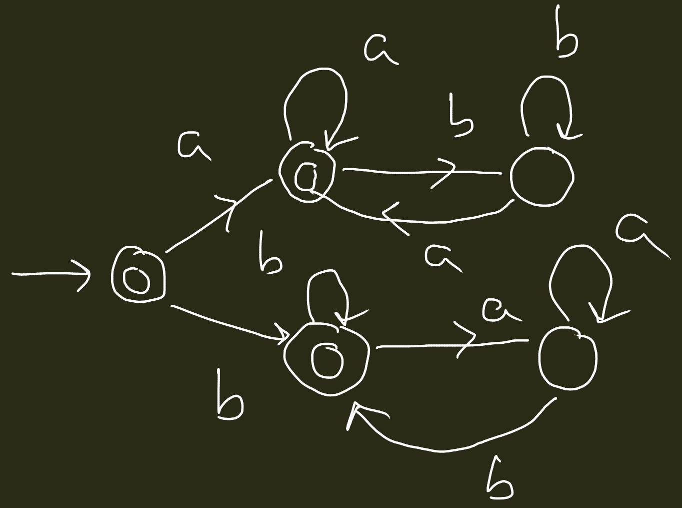
Take

$\underbrace{a^k}_{u}\ \underbrace{b^k}_{v}\ \underbrace{a^k b^k}_{w}$

3. Regular or not? Justify.

A = { w ∈ {a,b}* | w contains an equal number of occurrences of the substrings ab and ba }
B = { w ∈ {a,b}* | w contains an equal number of occurrences of the substrings aab and bba }

A is regular



B is not regular

$u = \cancel{(aab)}^k, \quad v = \cancel{(bba)}^k, \quad w = \epsilon$

$\underbrace{aabaab \ldots aab\,bba}_{u}\ \underbrace{bba \ldots bba}_{v}$

$x = \epsilon, \ y = b$

$u = (aab)^{k+1}\, bba$

$v = (bba)^k \quad w = \epsilon$

4. Let $A \subseteq N$ be a set of positive integers. Define

binary(A) = {binary representations of elements of A} $\subseteq \{0,1\}^*$, and
unary(A) = $\{0^n \mid n \in A\} \subseteq \{0\}^*$.

Prove/Disprove:

(a) If unary(A) is regular, then binary(A) is regular.
(b) If binary(A) is regular, then unary(A) is regular.

$A = \{2, 3, 5, 7, \ldots\}$

binary$(A) = \{10, 11, 101, 111, \ldots\}$

unary$(A) = \{00, 000, 00000, 0000000, \ldots\}$

010, 0010, ...

(b) $A = \{2^n - 1 \mid n \geq 1\}$

False

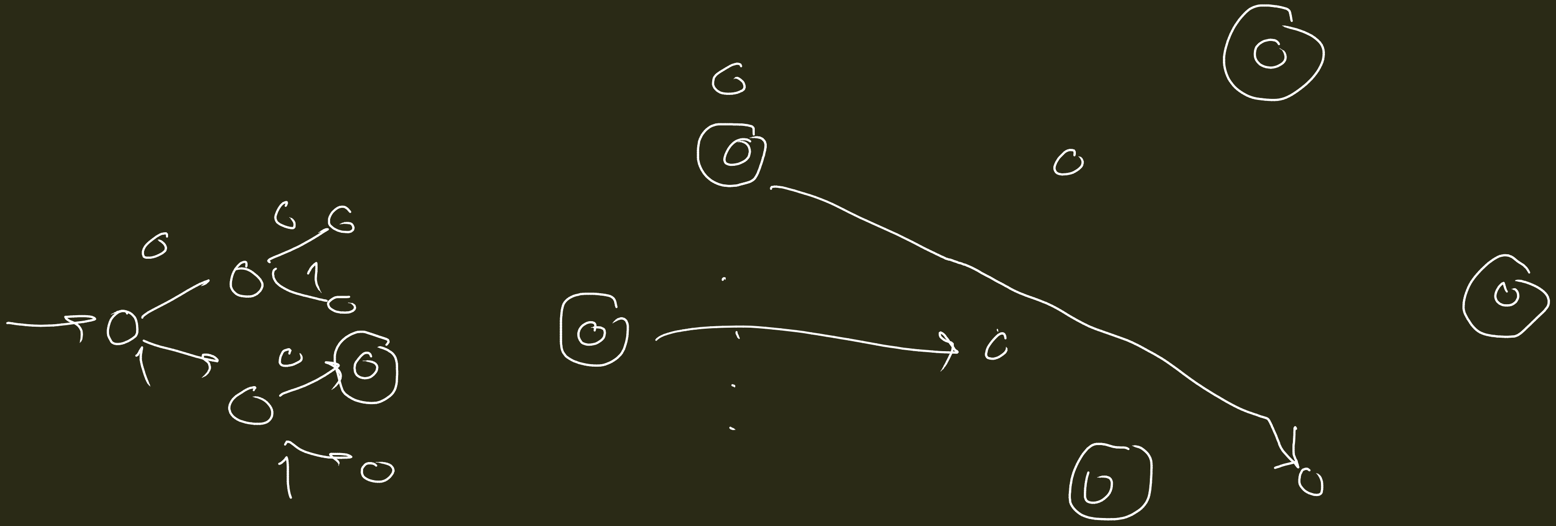
binary$(A) = \{1, 11, 111, 1111, \ldots\}$

lengths(unary(A)) is not u.p.
$= A$

(a) True

A is u.p.

$A = \{a_1, a_2, \ldots, a_k\} \cup \{n \equiv b_1, b_2, \ldots, b_\ell \pmod{p}, n \geq n_0\}$



(mod p) cycle

$\longleftarrow$ n_0 levels $\longrightarrow$

5. Prove that no infinite subset of { $a^n b^n$ | $n \geq 0$ } is regular.

$\underbrace{a^k}_{v} \; \underbrace{b^k}_{w}$

$u = \epsilon$

$\underbrace{a^{k'}}_{v} \; \underbrace{b^{k'}}_{w}$ for $k' \geq k$

$u = \epsilon$

6. Prove/Disprove: There exists a language L such that no infinite subset of L or its complement is regular.

$L = \{a^n b^n \mid n \geq 0\}$ ✓ ?

$\sim L$ contains $\mathcal{L}(a^+)$, $\mathcal{L}(b^+a^+)$

$\mathcal{L}((ab)^*(ab)^+)$

L	~L
0	
	1
2 - 3	
	4 - 7
8 - 15	
	16 - 31
32 - 63	
	64 - 127

7. Let D be a DFA with exactly k states and with $L = \mathscr{L}(D)$ infinite. Prove that L must contain a string of length in the range $[k, 2k)$. Give an example of D such that the shortest string in L of length $\geq k$ is of length $2k - 1$.

See Q4 of http://cse.iitkgp.ac.in/~abhij/course/theory/FLAT/Spring20/ct1.pdf

8. Let D be a DFA with exactly k states and with $L = \mathscr{L}(D)$ finite. Prove that the longest string in L must be of length $\leq k - 2$. Give an example of D where the longest string in L is of length equal to $k - 2$.