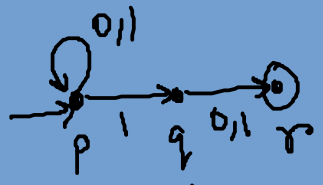
Tutorial 2-3

1. Start with the obvious NFA to accept all strings w over {0,1} such that the second last symbol of w is 1.
Use the subset construction procedure to convert this NFA to an equivalent DFA. Mark all the unreachable states (if any) in the converted DFA.



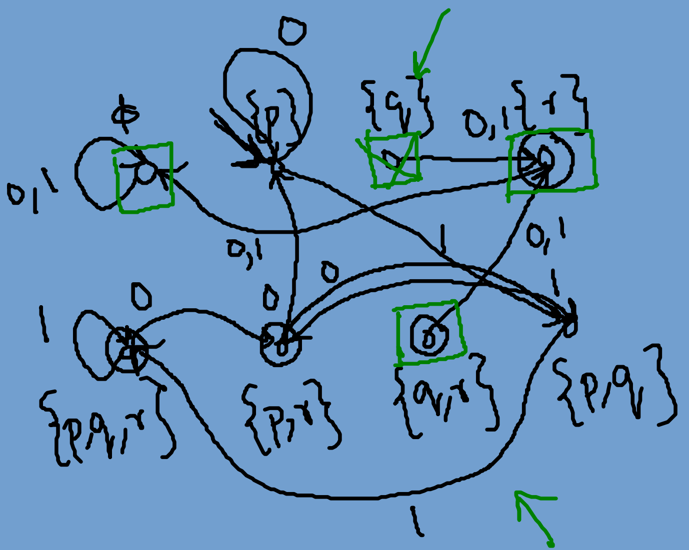
NFA

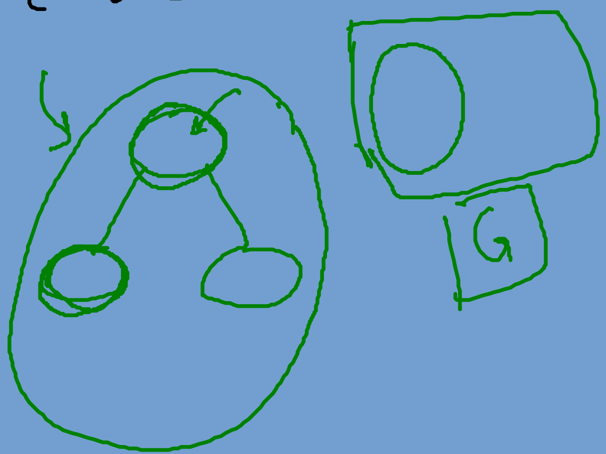
$$M = (Q, \{0,1\}, \delta, s, F)$$

$$Q = \{\phi, \{p\}, \{q\}, \{r\}, \{p,q\}, \{q,r\}, \{p,r\}, \{p,q,r\}\}.$$

$$s = \{p\}$$

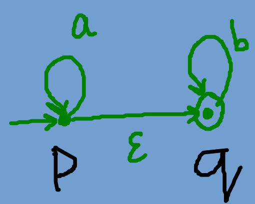
$$F = \{\{r\}, \{p,r\}, \{q,r\}, \{p,q,r\}\}$$





2. Consider the following NFA with ε-transition for the language

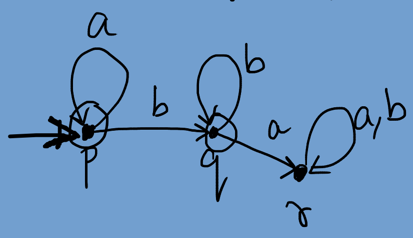
$$L=\{a^i b^j \mid i,j \geqslant 0\}$$



Convert to a DFA accepting the same language.

$$\Delta(p,\varepsilon)=\{p,q\}$$

$$\Sigma' = \Sigma \cup \{\varepsilon\}$$



$$L=\{\varepsilon\}\cup\{a^i \mid i\geqslant 1\}\cup\{b^j \mid j\geqslant 1\}$$
$$\cup\,\{a^i b^j \mid i,j\geqslant 1\}$$

3. Let k be a positive integer constant, and L = {w $\in$ {0,1}* | the k-th last symbol of w is 1}

(a) Prove that no DFA with less than 2^k states can accept L.

(b) Prove that no NFA with less than k+1 states can accept L.

(a) $|Q| < 2^k$. Consider all k-length strings in $\{0,1\}^*$.

PHP → There are 2 strings x, y ending at the same state p.

$\hat{\delta}(s,x) = \hat{\delta}(s,y) = p.$

$\hat{\delta}(s, x.0^{k-i-1}) = \hat{\delta}(s, y.0^{k-i-1}) = q.$

(b) Assume $|Q| \le k \rightarrow 2^k$ subsets of Q.

$x \in$ {k length strings in $\{0,1\}^*$.}

~~Assume~~ Suppose $\hat{\Delta}(S,x) = \phi$

$\rightarrow$ ✓ $x1^k$ $\quad \hat{\Delta}(S, x1^k) = \phi \; (\rightarrow\leftarrow)$

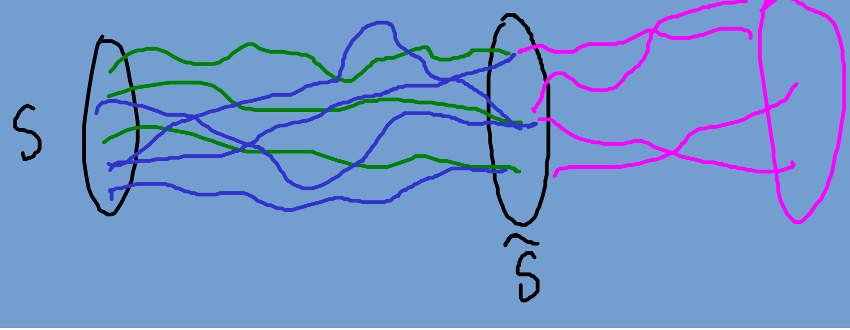
ϕ is an unreachable state.

$\le 2^k - 1$ reachable subsets of Q.

Consider the DFA with subset construction

x $\quad \hat{\Delta}(S,x) = \hat{\Delta}(S,y) = \hat{S}$

y $\rightarrow \forall z \quad \hat{\Delta}(S,xz) = \hat{\Delta}(S,yz)$



4. Let A and B be regular languages. Prove that the following language is also regular.

$$A \Delta B = (A - B) \cup (B - A)$$

$A - B = A \cap \neg B$ $\quad\quad$ $B - A = B \cap \neg A$.

$$A \Delta B = \underbrace{\underbrace{(\underline{A} \cap \neg \underline{B})}_{} \cup \underbrace{(\underline{B} \cap \neg \underline{A})}_{}}_{}$$

Σ $A = A' \subseteq \Sigma^*$ s.t no FA exists for A'

$\Sigma^* - A'$

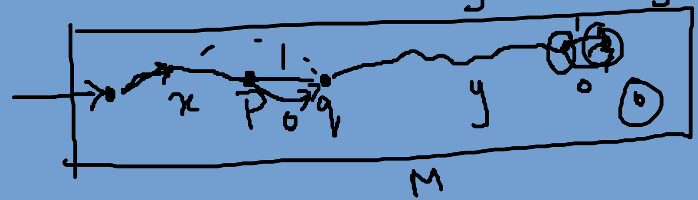
$A \cup B$ regular Σ^*
A regular A'
B regular? Σ^*

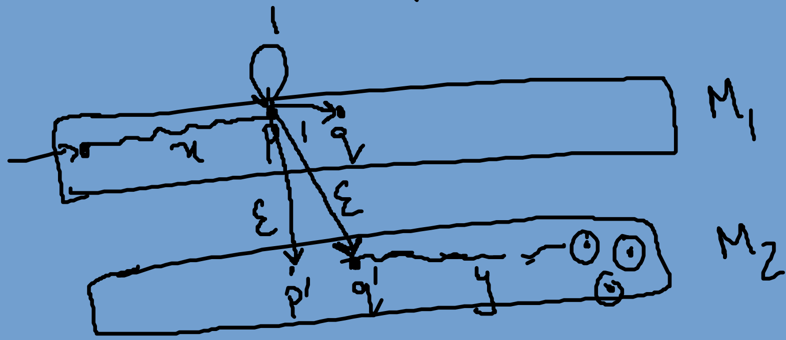
5. Let A be a regular language over {0,1}.
(a) Prove that the language B = $\{xy \in \{0,1\}^* \mid x1y \in A\}$ is regular.

$x0y \quad x11y$

$1111 \in A$
$111 \in B$

$(A \cup B) \triangle A$
$= ((A \cup B) - A) \cup (A - (A \cup B))$
$= (B - A)$
$= B - A$





$\Rightarrow z \in B \quad xy \in B$
$\exists\, x1y \in A$
$\exists\, s, s_1 \ldots s_i\, s_{i+1} \ldots s_n$ in M s.t $s_n \in F$.
Consider $s, s_1 \ldots s_i, s'_{i+1} \ldots s'_n \quad s'_n \in F_{M'}$

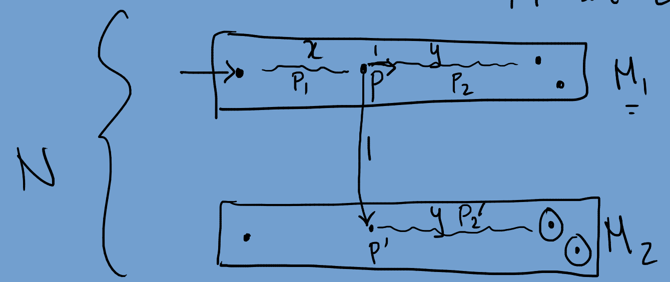
$\Leftarrow$ z is accepted
$\exists$ an ε-transition from $M_1 \to M_2$
$p \to q' \qquad \delta_M(p, 1) = q$

Acceptance: $s\, s_1 \ldots p, q' \ldots f' \to xy$
$s\, s_1 \ldots p, q \ldots f \to$ accepting $x1y$ in M.

(b) Prove that $C = \{x1y \in \{0,1\}^* \mid xy \in A\}$ is regular.

$\exists z \in A$
$z = xy$
$z' \in C = x1y$

M s.t $L(M) = A$



($\Rightarrow$) If $xy \in A$ then $x1y \in L(N)$:
DFA M has a unique path labelled xy : $s, s_1, .., p, \quad s_n \in F$
$\hat{\delta}(s, x) = p \quad , \quad \hat{\delta}(p, y) = s_n$.

In N, $\exists$ $x1y$-path $s \cdots p \xrightarrow{1} p' \cdots s_n'$.

– ($\Leftarrow$) Consider $z \in L(N)$. From $s \longrightarrow$ some s_n' $\exists$ exactly 1 new 1-transition from $M_1 \to M_2$ of the form $p' \in \Delta(p, 1)$

$\Rightarrow$ Path will be of form $s \overset{P_1}{\sim} p \xrightarrow{1} p' \overset{P_2'}{\cdots} s_n'$

Consider $P_1 \cdot P_2 \to$ Path labelled by xy in M_1

$\therefore$ $xy \in A$.

6. Let L be a regular language over Σ.

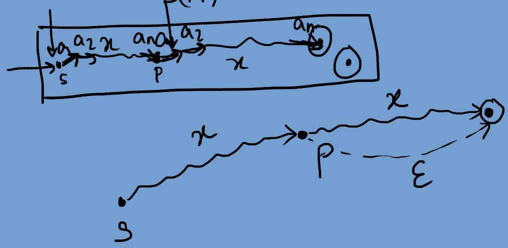
(a) Prove that the following language is also regular.

$$L' = \{x \in \Sigma^* \mid xx \in L\}$$

abab $\in L$

ab $\in L'$

$L(M) = L$



$\hat{\delta}(s, x) = p$

$\hat{\delta}(p, x) = f \in F$

? shortest s-p path

Algorithm:

1. For each state p, let $\hat{\delta}(s, x) = p$
2. Consider $q = \hat{\delta}(p, x)$.
3. If $q \in F$ then include $p \in F$.

$$L^p = \{x \in \Sigma^* \mid xx \in L, \hat{\delta}(s, x) = p\}$$

$$L' = \bigcup_{p \in Q} L^p.$$

$\{(p, q) \mid q \in F\}$.

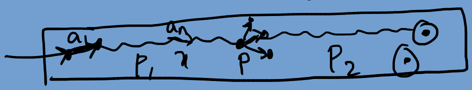
$$M' = (Q \times Q, \Sigma, \delta', (s, p), F')$$

$$\delta'((p, q), a) = (\delta(p, a), \delta(q, a))$$

(b) Prove that the following language is also regular

$$L' = \{x \in \Sigma^* \mid xy \in L \text{ for some } y \in \Sigma^* \text{ with } |x| = |y|\}$$

$M = (Q, \Sigma, \delta, s, F)$



$\hat{\delta}(s,x) = p$ $\quad |x| = |y|$

$\exists y : \hat{\delta}(p,y) \in F$

For all $p \in Q$,

Claim: $L^p = \{x \in \Sigma^* \mid x \in L' \text{ and mid-pt. is } p\}$. is regular.

Implication: $L' = \bigcup_{p \in Q} L^p$ is regular.

Pf of: NFA $N = (Q \times Q, \Sigma, \Delta', S', F')$

a

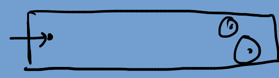
$\rightarrow \Delta'((q',q), a) = \left[\bigcup_{\substack{\text{transitions } a_i \\ \text{out of } q}} (\delta(q',a), \delta(q,a_i)) \right]$

$\rightarrow S' = \{(s,p)\}$

$\rightarrow F' = \{(p,q) \mid q \in F\}$

(c) Prove that the following language is also regular.

$L = L(M)$



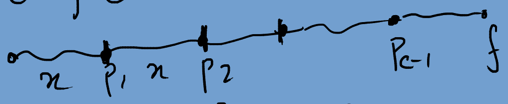
M

$(Q, \Sigma, \delta, s, F)$ → k states in Q.

$$L' = \{ x \in \Sigma^* \mid x^n \in L \text{ for some } n \geq 1 \}$$

$L^c = \{ x \in \Sigma^* \mid x^c \in L \}$ is regular →

Similar as 6(a).
c-fold construction



$L^{p_1, p_2 \cdots p_{c-1}} \subseteq L^c$ and $p_1 \cdots p_{c-1}$ are the break pts.

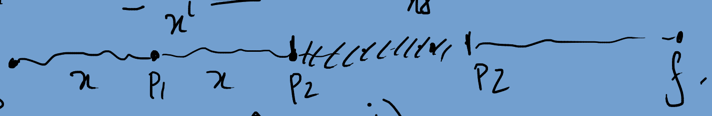
$$L^c = \bigcup_{p_1 \cdots p_{c-1} \in Q} L^{p_1, \cdots p_{c-1}}$$

Cl: If $x \in L'$, $\exists n \leq k$ s.t $x^n \in L$.

pf: If $x \in L'$, $\exists t$ s.t $x^t \in L$.

If $t \leq k$ done.

O/w. $t \geq k+1$. (minimum such t)



$\hat{\delta}(s, x) = p_1$, $\hat{\delta}(s, x^i) = p_i$.

$\exists i < j$ s.t $\hat{\delta}(s, x^i) = \hat{\delta}(s, x^j)$.

$\hat{\delta}(s, x^{t-j+i}) = \hat{\delta}(s, x^t)$

$t-j+i < t$ contradicting minimality of t.

7. Prove or disprove:

(a) $(0+1)^* \equiv 0^* + 1^*$ False. 010 on LHS but not RHS.

(b) $(0^*1)^* \equiv (0^*1^*)^*$

Any string of the form 0^*1^*0 is in LHS but not RHS.

False 010 will not be in LHS

8. $\alpha = (a+b)^* \underline{ab} (a+b)^*$

Give regular expressions for $\sim\alpha$ if

→ (a) $\Sigma = \{a, b\}$

(b) $\Sigma = \{a, b, c\}$

$L(\alpha) = L(a^*) + L(b^*) + \{x \in \{a,b\}^* \mid$ all b's appear before a's$\}$

↓

$b^* a^*$.

(a) $L(\sim\alpha) = \sim L(\alpha) = b^* a^*$.

$a^* + b^* + b^* a^* \equiv b^* a^*$.

(b) $L(\sim\alpha) =$ Anything containing c $\cup\, L(b^* a^*)$ $-$ $\underline{@c@} = (a+b+c)^* c (a+b+c)^*$

$\sim\alpha \equiv (a+b+c)^* c (a+b+c)^* + b^* a^*$.