

**SOME DECIDABLE PROBLEMS
ABOUT TURING MACHINES**

**Abhijit Das
Sudeshna Kolay**

Department of Computer Science and Engineering
Indian Institute of Technology Kharagpur

March 29, 2022

Properties of RE Sets / Turing Machines?

M accepts ε

- This is a property of RE sets.
- This is specified by any member of $\{M \mid \varepsilon \in \mathcal{L}(M)\}$.
- This is a non-trivial property, and is undecidable by Rice's theorem, Part 1.
- The property is monotone, so Rice's theorem, Part 2, is not applicable.
- The property is semidecidable.
 - Simulate M on ε , and accept if and only if M accepts.
 - For any M with $\varepsilon \in \mathcal{L}(M)$, the simulation accepts and halts.
 - For an M with $\varepsilon \notin \mathcal{L}(M)$, the simulation may halt in the reject state, or loop.

Properties of RE Sets / Turing Machines?

M halts on ε

- This is not a property of RE sets.
- If $\varepsilon \notin \mathcal{L}(M)$, then upon input ε , M may have a choice to halt after rejecting or loop.
- Rice's theorems cannot be applied in this context.
- The language $\{M \mid M \text{ halts on } \varepsilon\}$ is r.e. but not recursive.
- RE: Simulate M on ε , and accept if and only if the simulation halts.
- Not recursive: Reduction from HP: $M \# w \mapsto N$.
 - Simulate M on w .
 - If the simulation halts, accept.

Problem 1

Given M , decide whether M contains at least 2022 states.

- A Turing machine looks at the encoding of M , and finds out the answer.
- This machine runs in finite time for every input.

Problem 2

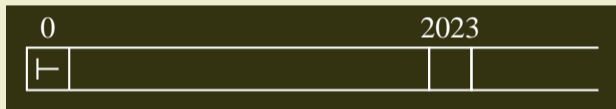
Given M , decide whether M halts within 2022 steps on input ε .

- Simulate M on ε for (at most) 2022 steps.
- If the simulation halts (after accepting/rejecting), accept.
- If the simulation does not halt after 2022 steps, reject.
- This machine is also a decider.

Problem 3

Given M , decide whether M takes more than 2022 steps on some input.

- M takes more than 2022 steps on some input \iff
 M takes more than 2022 steps on some input of length ≤ 2022 .
- Suppose that M takes ≤ 2022 steps on all inputs of length ≤ 2022 . Supply an input w of length > 2022 to M .



- Within 2022 steps, M cannot see more than 2022 symbols from the input.
- This initial behavior of M on w is the same as its behavior on the prefix of w of length 2022. M is deterministic. M halts on w within 2022 steps.
- A decider simulates M on all inputs of length ≤ 2022 , each for 2022 steps.
- If some simulation takes more than 2022 steps, accept, else reject.

Problem 4

Given M , decide whether M takes more than 2022 steps on all inputs.

- M takes more than 2022 steps on all inputs \iff
 M takes more than 2022 steps on all inputs of length ≤ 2022 .
- It suffices to simulate M on all inputs of length ≤ 2022 , each for 2022 steps.

Problem 5

Given M , decide whether M ever moves to the right of the 2022-nd cell on input ε .

- This problem is certainly semidecidable: Simulate M on ε .
- It is decidable too.
- Let $m = |Q|$ (number of states).
- Let $k = |\Gamma|$ (number of symbols in the tape alphabet).
- Suppose M never goes to the right of the 2022-nd cell.
- Total number of configurations possible is $2023mk^{2022}$.
- Simulate M on ε for at most $2023mk^{2022}$ steps.
- If the head ever moves to the right of the 2022-nd cell during the simulation, accept.
- If the simulation halts without the head moving to the right of the 2022-nd cell, reject.
- Otherwise, some configuration is repeated (pigeon-hole principle).
- Thus the machine must have entered an infinite loop, and will never go beyond the 2022-nd cell. Reject.

Three Possibilities

Rice's theorems do not apply to problems specific to Turing machines but not to their languages.

- The problem may be decidable
 - Design a decider for the problem.
 - You must prove that on all inputs, the decider halts after making correct accept/reject decisions.
- The problem may be semidecidable but not decidable.
 - Design a Turing machine (not total) to semidecide the problem.
 - You must prove that all accept decisions are made in finite amounts of time.
 - Use reduction from an undecidable problem like HP.
- The problem may be not even semidecidable.
 - Use reduction from a non-semidecidable problem like \overline{HP} .

Tutorial Exercises

1. Prove that the following problems on a TM M are decidable.
 - (a) Decide whether M halts on some input within 2021 steps.
 - (b) Decide whether M halts on all inputs within 2021 steps.
 - (c) Decide whether M runs for at least 2021^{2021} steps for input a^{2021} .
 - (d) Decide whether M on input ε moves left at least ten times.
 - (e) Decide whether M on a given input w moves left at least ten times.
2. Is the problem whether a Turing machine on any input reenters the start state decidable or not? Prove.
3. Input: A Turing machine M . Decidable/Semidecidable/Not? Prove.
 - (a) M halts on exactly 2021 input strings.
 - (b) M halts on at least 2021 input strings.
4. Input: Two Turing machines M and N . Decidable/Semidecidable/Not? Prove.
 - (a) M takes more steps than N on input ε .
 - (b) M does not take more steps than N on input ε .