# NONDETERMINISTIC

# TURING MACHINES

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# Nondeterminism

- By definition, Turing machines are deterministic.
- TM = DTM = Deterministic Turing machine.
- NTM = Nondeterministic Turing machine.
- An NTM has $\delta(p,a) = \{(q_1,b_1,d_1),(q_2,b_2,d_2),\ldots,(q_k,b_k,d_k)\}$ for some $k \geqslant 0$.
- If $k = 0$, the machine gets stuck.
- If $k \geqslant 2$, the machine chooses one of the $k$ moves nondeterministically.
- The maximum value of $k$ over all combinations of $p$ and $a$ is called the (maximum) **fanout**. Denote the maximum fanout by $f$.
- The machine accepts if and only if **some** sequence of nondeterministic choices lets the machine reach the accept state.
- Other nondeterministic choices may lead to the stuck or a looping condition or even to the reject state.

# Computation Histories

- Each sequence of choices gives a computation history.
- A history may terminate in the accept/reject state or in any other state in the stuck condition.
- Looping leads to an infinite history.
- The NTM **accepts** an input if and only if there is a finite computation history ending in the accept state.
- **Explicit reject:** All computation histories are finite, and none ends in the accept state.
- **Implicit reject:** No accepting history along with one or more infinite histories.
- A specific reject state is not needed for an NTM (even if total).
- The language of an NTM is the set of all input strings it accepts.
- An NTM is called **total** if all histories on all inputs are finite.
- All histories start with the initial configuration, and form a (potentially infinite) computation tree of configurations.

# A Nondeterministic Compositeness Test

- A two-tape NTM for $\{a^n \mid n \text{ is composite}\}$.
- Input $a^n$ is supplied to the first tape.
- **Initial check:** Try to copy two $a$'s from the first tape to the second. If the attempt fails ($n = 0, 1$) or if no other symbol is left in the input ($n = 2$), reject and halt.
- If the machine is here, both the heads are pointing to the third cell (third $a$ in the first tape and the first blank cell in the second tape).
- **Non-deterministic copying stage:** If there are more $a$'s left in the input, make a non-deterministic choice.
  - Copy another $a$ to the second tape, and move both the heads right.
  - Go to the division stage.
- **Division stage:** Suppose $d$ number of $a$'s are copied to the second tape.
  - Check whether the first head points to a blank cell ($d = n$). If so, reject and halt.
  - Otherwise, repeatedly subtract $d$ from $n$. If some subtraction run erases the entire first tape, accept and halt. If some subtraction run prematurely ends, reject and halt.

# Encoding Configurations

- The future work of a Turing machine (DTM/NTM) depends on:
  - State of the finite control.
  - Content of the tape.
  - The position of the head.
- A configuration is specified as the string $(p, u\underline{a}v)$, where $p \in Q$, $u, v \in \Gamma^*$, and $a \in \Gamma$. The underline represents the position of the head.
- We assume that $Q \cap \Gamma = \emptyset$.
- Then the configuration can be encoded as the string $upav \in (Q \cup \Gamma)^*$.
- The initial configuration on input $w$ is $s \vdash w$.
- An accepting configuration is $utav$ for any $u, a, v$.

# Simulation of an NTM by a DTM

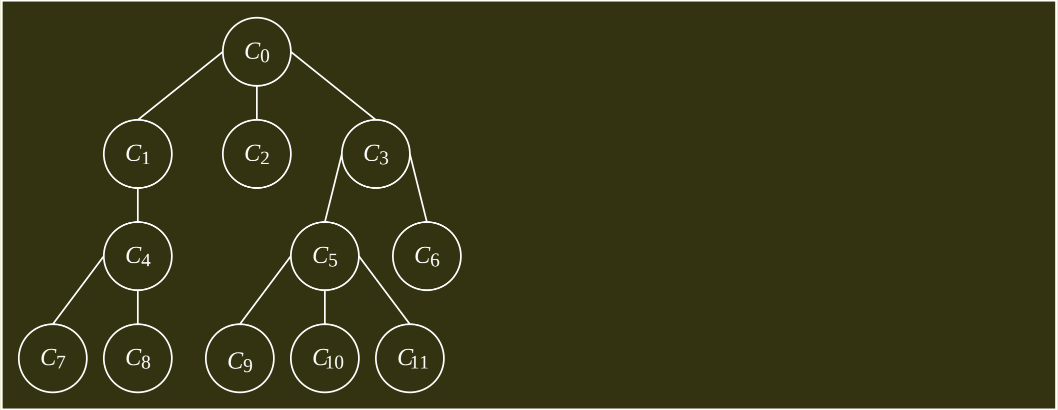
**Theorem**

*For every NTM N, there exists a DTM D with* $\mathscr{L}(D) = \mathscr{L}(N)$.

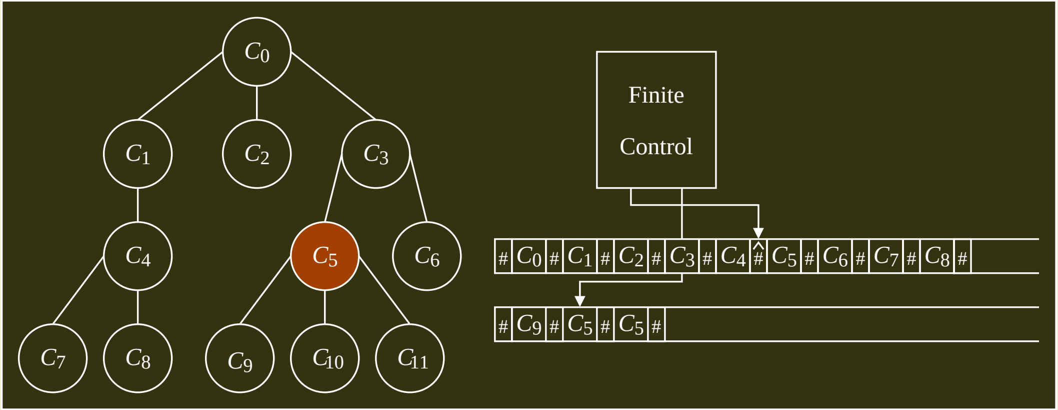
**Theorem**

*For every total NTM N, there exists a total DTM D with* $\mathscr{L}(D) = \mathscr{L}(N)$.

# Simulation of an NTM by a DTM



# Simulation of an NTM by a DTM



# Simulation of an NTM $N$ by a DTM $D$

- $D$ has the information of $N$ in its finite control.
- $D$ is a two-tape machine.
  - The first tape is used to implement a queue of configurations of $N$.
  - The second tape is used to compute next configurations of $N$.
  - The configurations are separated by a separator #.
  - One of the configurations is active denoted by a marker on the preceding #.
- The input $w$ of $N$ is given on the first tape of $D$.
- $D$ uses the second tape to generate $C_0 = s \vdash w$.
- $D$ replaces $w$ by $\hat{\#}C_0\#$ on the first tape.
- $D$ then enters a loop.

# Simulation of an NTM $N$ by a DTM $D$

- $D$ locates the current active configuration $C_i$.
- If there is no active configuration, $D$ rejects and halts.
- Otherwise, $D$ finds from $C_i$ the state $p$ of $N$ and the tape symbol $a$ scanned by the head of $N$.
- If $p = t$, $D$ accepts and halts.
- $D$ consults the transition table of $N$ to identify

$$\delta_N(p,a) = \{(q_1,b_1,d_1),(q_2,b_2,d_2),\ldots,(q_k,b_k,d_k)\}.$$

- $D$ makes $k$ copies of $C_i$ to the second tape.
- $D$ converts each copy by a transition possibility.
- $D$ then copies the new configurations at the end of the first tape.
- Finally, $D$ moves the active marker from the current # to the next #.

# More about the Simulation

- $D$ makes a breadth-first traversal in the computation tree of $N$.
- If $N$ is total, then $D$ is total too.
- Let $n$ be the number of steps in the longest computation history of $N$ on some input.
- Let $f \geqslant 2$ be the maximum fanout of $N$.
- $D$ needs to generate at most $1 + f + f^2 + \cdots + f^n = \frac{f^{n+1}-1}{f-1}$ configurations of $N$.
- The running time of $N$ is taken as $n$.
- The running time of $D$ is $\mathrm{O}(lf^n)$, where $l$ is the longest configuration of $N$.
- It is not known whether the exponential slowdown of the simulation can be avoided in all cases.

# Tutorial Exercises

1. Determine the (nondeterministic) running time of the compositeness test.

2. Why cannot the nondeterministic compositeness be used to accept the language $\{a^p \mid p \text{ is a prime}\}$?

3. Design an NTM to accept each the following languages.

   (a) $\{ww \mid w \in \{0,1\}^*\}$.
   (b) $\{wxyxz \mid w,x,y,z \in \{0,1\}^*,\ |x| = 100\}$.
   (c) $\{wxyxz \mid w,x,y,z \in \{0,1\}^*,\ |x| \geqslant 100\}$.

4. Suppose that the simulator $D$ uses its first tape as a stack of configurations of $N$. Will the simulation work? Justify.

5. A TM $M$ has a two-way infinite tape. Initially, all cells on the tape are blank. Only one cell is storing the symbol #. The head of $M$ is pointing to a blank. The task of $M$ is to locate the cell storing #. Propose a strategy for doing this

   (a) if $M$ is an NTM, and
   (b) if $M$ is a DTM.
   (c) Compare the nondeterministic and deterministic running times.