Pattern Matching and Regular Sets

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- Example: When we type *.ext on a console we are pattern matching with any file with the same extension.
- Note: Pattern matching is an important application of finite automata. Grep, fgrep, egrep are pattern matching commands and they use finite automata in their implementation.

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- Two kinds – *atomic and compound*.
- Notational Convention: Denoted by Greek letters α, β etc.

Atomic Patterns

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- Given a pattern α, $L(\alpha) = \{x | x \text{ matches the pattern } \alpha\}$.
- What are the strings that match to these atomic patterns?
  $\{a\}, \{\epsilon\}, \emptyset, \Sigma, \Sigma^*$, respectively.

Compound Patterns

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- Inductively defined from atomic patterns using **binary operators** $+, \cap, \cdot$, and **unary operators** $^*, ^+, \sim$ (or $\neg$).
- If α and β are patterns then so are $\alpha + \beta$, $\alpha \cap \beta$, $\alpha \cdot \beta$, α^*, α^+, $\sim \alpha$ (or $\neg\alpha$).

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- $L(\sim \alpha) = \sim L(\alpha) = \Sigma^* - L(\alpha)$

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- Above language L if $\Sigma = \{a, b\}$: what is the pattern?
  $\epsilon + @b$.

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3. Patterns α and β are equivalent ($\alpha \equiv \beta$) if $L(\alpha) = L(\beta)$. How can you find out equivalence?
4. Which operators are redundant?

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- Eg. ϵ is equivalent to $\sim(\#@)$ or $\emptyset^*$.
  @ is same as $\#^*$.
  $^+$ not necessary: $a^+ = aa^*$
  $\#$ not necessary: $\Sigma = \{a, b, ..z\}$ means $\# = a + b + \ldots z$
  $\cap$ is redundant – $a \cap b = \sim(\sim a + \sim b)$
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- Thus, each pattern is equivalent to one with only atomic patterns $a \in \Sigma, \epsilon, \emptyset$ and operators $+, \cdot, ^*$.
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- A pattern that only uses the above atomic patterns and operators is called a **regular expression**.

Notational Conventions for Patterns

- $\cdot$ given preference over $+$.
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- Or use parenthesis properly!

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- **Theorem**: Equivalent statements:
- A. A is a regular set
- B. $A = L(\alpha)$ for a pattern α
- C. $A = L(\alpha)$ for a regular expression α.

Theorem: $C \implies B$

$A = L(\alpha)$ for a regular expression $\alpha \implies A = L(\alpha)$ for a pattern α.
Proof: $C \implies B$ from definition.

Theorem: $B \implies A$

$A = L(\alpha)$ for a pattern $\alpha \implies A$ is a regular set

- Proof :

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- Proof :
- Singleton set $\{a\}$ is regular (How?)
- $\{\epsilon\}$ is regular (How?)
- $\emptyset$ is regular (How?)

Theorem: $B \implies A$ contd.

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- Now we induct on the length of the pattern. What is the form of the pattern?
- Base case:
  1. a for some $a \in \Sigma$: $L(a) = \{a\}$ a regular set
  2. ϵ : $L(\epsilon) = \{\epsilon\}$ a regular set
  3. $\emptyset$: $L(\emptyset) = \emptyset$ a regular set
  4. $\#$ - redundant
  5. @ – redundant

Theorem: $B \implies A$ contd.

- Induction: For compound pattern, induction on the number of operators.

  6. β^+ – redundant

  7. $\beta + \gamma$: $L(\beta + \gamma) = L(\beta) \cup L(\gamma)$. By induction, β and γ give regular sets. Closure under $\cup$ gives regular set.

  8. $L(\beta \cap \gamma) = L(\beta) \cap L(\gamma)$: regular set

  9. $L(\beta \cdot \gamma) = L(\beta) \cdot L(\gamma)$: regular set

  10. $L(\beta^*) = L(\beta)^*$: regular set

  11. $L(\sim \beta)$ or $L(\neg\beta) = \sim L(\beta)$: regular set

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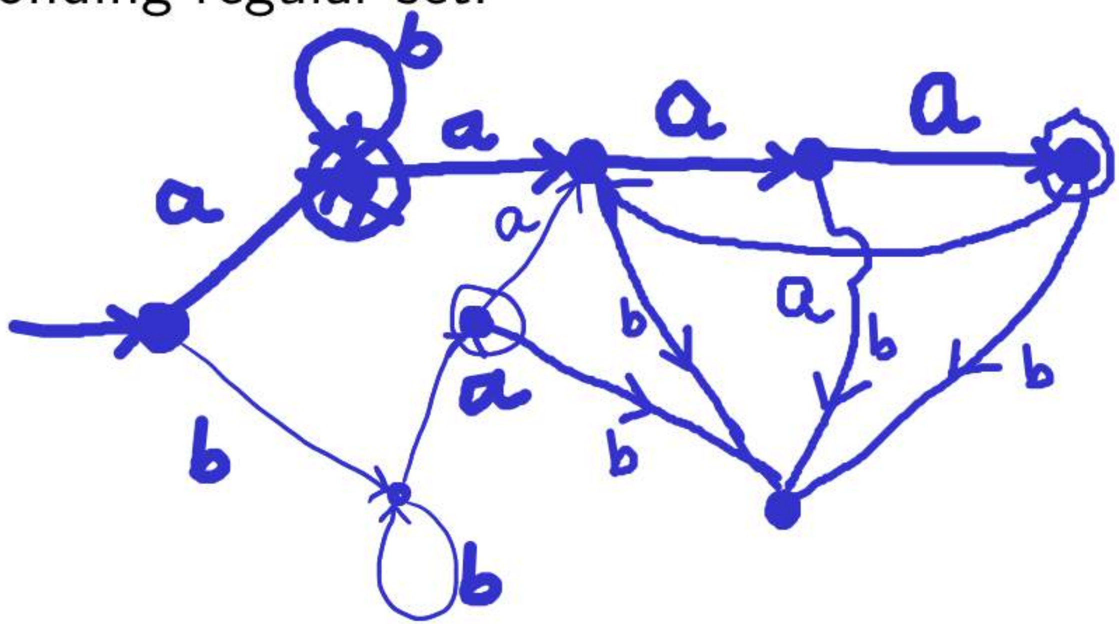
- Thus, done.

Intermezzo

- Eg: Convert the regular expression $(ab^* + b^*a)(aaa)^*$ to the corresponding regular set.



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- Going back to the Questions: Q1 – How will you match a string to a given pattern? [$B \implies A$]
- (Q3 – We will have a look later).

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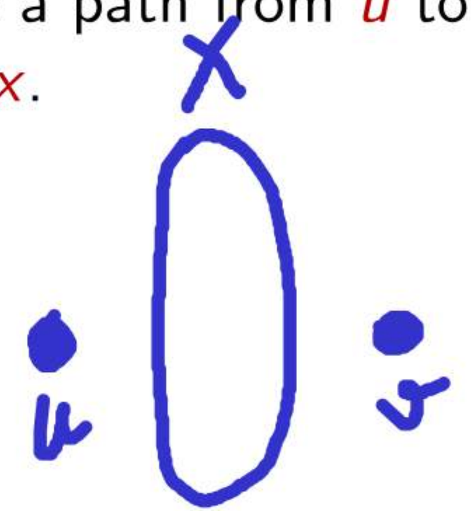

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- **Aim**: For a subset $X \subseteq Q$ and states u, v, let α_{uv}^{X} be a regular expression for all strings x that have a path from u to v with all internal vertices in X labelled by x.
- **Implication**: If we did this for all u, v and all subsets X, then $\Sigma_{s \in S} \Sigma_{f \in F} \alpha_{sf}^{Q}$ would be a regular expression for all strings in $L(M)$. We will be done.

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- If $u \neq v$, $\alpha_{uv}^{\emptyset}$
  $=$ Sum over all elements in Σ', Σ' = {alphabets that are labels on outgoing edges of u}
  $= \emptyset$ if $\Sigma' = \emptyset$
- If $u = v$, then $\alpha_{uv}^{\emptyset}$
  $=$ Sum over all elements in $\Sigma' + \epsilon$ [all possible labelled loops plus staying in the same state means no input read]
  $= \epsilon$ if $\Sigma' = \emptyset$

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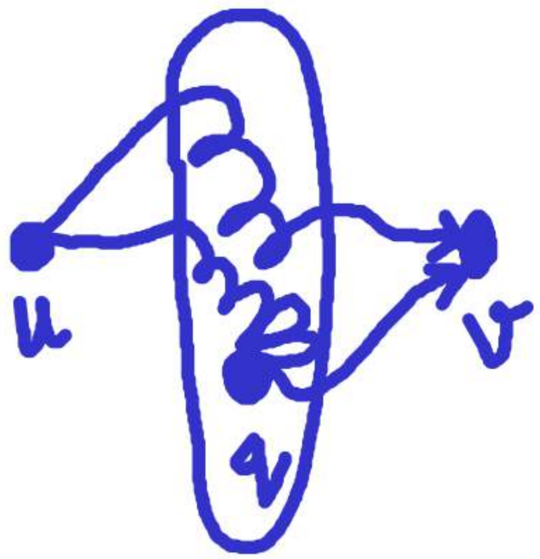

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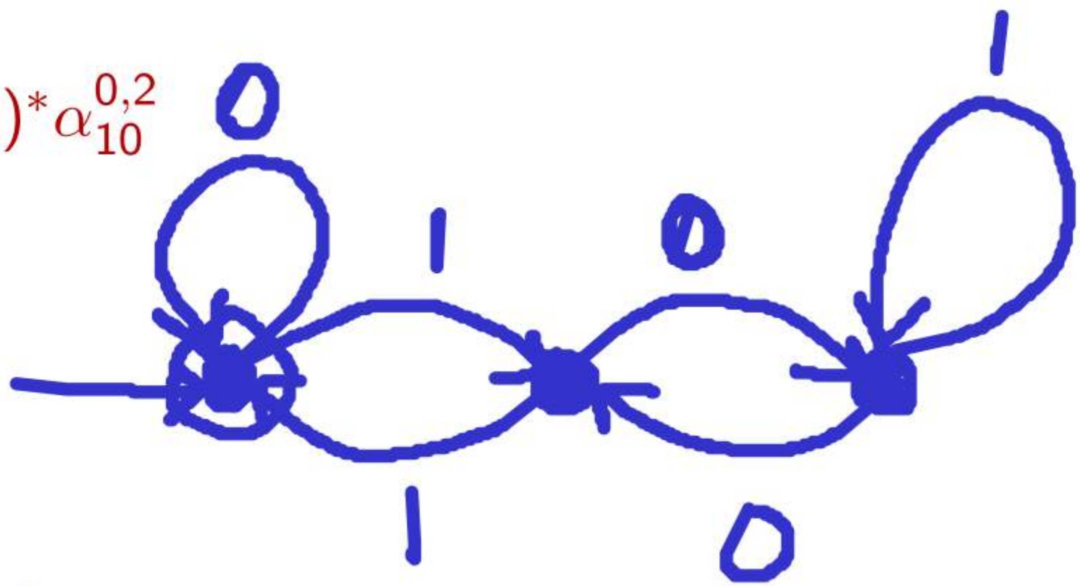
- By IH on size of X, RHS combines to form a regular expression. So, we are done with proving Aim, and therefore Implication.

Example: Regular Expressions for DFA for binary strings divisible by 3

- $\alpha_{00}^{0,1,2} = \alpha_{00}^{0,2} + \alpha_{01}^{0,2}(\alpha_{11}^{0,2})^*\alpha_{10}^{0,2}$
- $\alpha_{00}^{0,2} = 0^*$
- $\alpha_{01}^{0,2} = 0^*1$
- $\alpha_{11}^{0,2} = 01^*0 + 10^*1$
- $\alpha_{10}^{0,2} = 10^*$
- So $0^* + 0^*1(01^*0 + 10^*1)^*01^*$



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- Q3 is asking to solve an NP-hard problem!

Laws of Simplification

- $\alpha + (\beta + \gamma) \equiv (\alpha + \beta) + \gamma$
- $\alpha + \beta \equiv \beta + \alpha$
- $\alpha + \emptyset \equiv \alpha$
- $\alpha + \alpha \equiv \alpha$
- $\alpha(\beta \cdot \gamma) \equiv (\alpha \cdot \beta)\gamma$
- $\epsilon \cdot \alpha \equiv \alpha \cdot \epsilon \equiv \alpha$
- $\alpha(\beta + \gamma) \equiv \alpha \cdot \beta + \alpha \cdot \gamma$

Laws of Simplification

- $(\alpha + \beta)\gamma \equiv \alpha \cdot \gamma + \beta \cdot \gamma$
- $\emptyset \cdot \alpha \equiv \alpha \cdot \emptyset \equiv \emptyset$
- $\epsilon + \alpha \cdot \alpha^* \equiv \alpha^* \equiv \epsilon + \alpha^* \alpha$

Notation: $\alpha \leq \beta \iff L(\alpha) \subseteq L(\beta) \iff L(\alpha + \beta) = L(\beta)$, or $\alpha + \beta \equiv \beta$

- $\beta + \alpha \cdot \gamma \leq \gamma \implies \alpha^* \beta \leq \gamma$ (show set theoretically)
- $\beta + \gamma . \alpha \leq \gamma \implies \beta . \alpha^* \leq \gamma$ (show set theoretically)

Other Equations

- $(\alpha \cdot \beta)^* \alpha \equiv \alpha(\beta \cdot \alpha)^*$: Argue that $(\alpha \cdot \beta)^i \cdot \alpha \equiv \alpha(\beta \cdot \alpha)^i$
- $(\alpha^* \beta)^* \alpha^* \equiv (\alpha + \beta)^*$
- $\alpha^*(\beta \cdot \alpha^*)^* \equiv (\alpha + \beta)^*$: Same as above if α^* is taken as γ and the first equation is applied.
- $(\epsilon + \alpha)^* \equiv \alpha^*$: Substitute appropriately in 2nd equation
- $\alpha \cdot \alpha^* \equiv \alpha^* \cdot \alpha$: assume α is not ϵ as otherwise it trivially follows.
  Neither LHS nor RHS matches with the ϵ string. Add ϵ to both sides and this gives α^* to both sides, so LHS must be same as RHS.

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- $\equiv ((\epsilon + 0)(\epsilon + 0)1)^*(\epsilon + 0)(\epsilon + 0)$
- This defines a set of strings that do not have more than 2 consecutive 0's in any substring.