Finite Automaton and Regular sets

Machine Model Parlance

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- System with *finite states and transitions* among them – finite state transition system. This leads to the math model called **finite automaton**.

Deterministic Finite Automaton: Math representation

DFA: Structure $M = (Q, \Sigma, \delta, s, F)$
Q – finite set of states
Σ – Finite alphabet
δ – $Q \times \Sigma \to Q$, state transition function. Which state to move to in response to input alphabet. Finite set of ordered pairs.
s – start state
F – subset of Q called accept/final states.
Note: A DFA structure can be represented as a finite string – pick favourite encoding. The structure definition involves finite items, so the structure can be represented as a finite string.

An Example DFA

$$M = (Q, \Sigma, \delta, s, F)$$

$$Q = \{0, 1, 2\} \quad , \quad \Sigma = \{a, b\}$$

$$\delta: \quad \delta(0, a) = 1 \;,\; \delta(0, b) = 0$$

$$\delta(1, a) = 2 \;,\; \delta(1, b) = 1$$

$$\delta(2, a) = \delta(2, b) = 2$$

$$s = 0 \quad , \quad F = \{2\}$$

Transition Table

$\delta: \delta(0,a)=1, \delta(0,b)=0, \delta(1,a)=2, \delta(1,b)=1, \delta(2,a)=\delta(2,b)=2$

$s=0, \quad F=\{2\}$

	a	b
→0	1	0
1	2	1
2F	2	2

Transition Diagram

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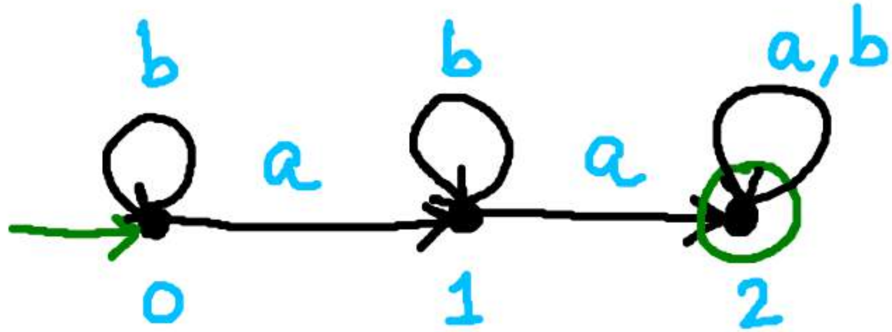

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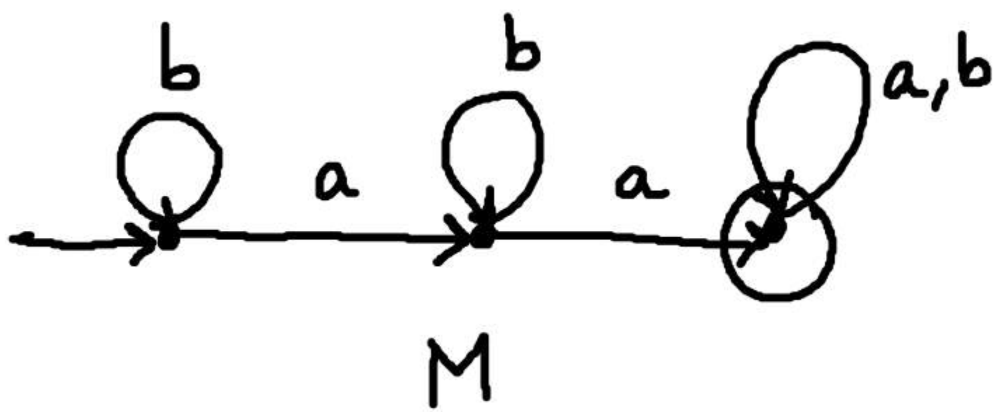
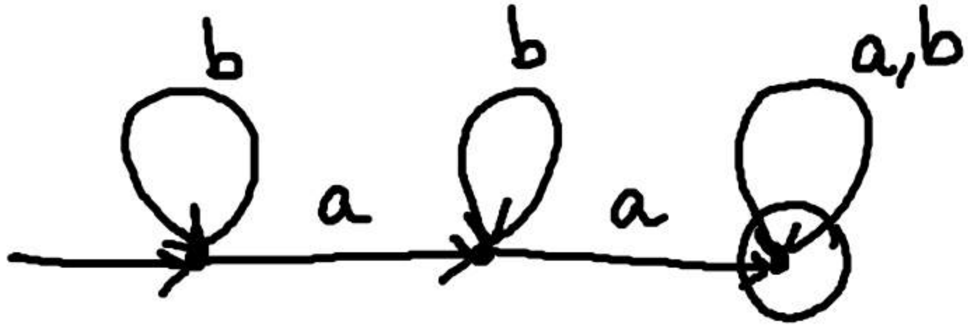
- **Ex**: What are the strings accepted in the example above?



Strings accepted by Example DFA



Accepted strings: Must have at least 2 a's

babbb ✗

baab ✓

abbbbab ✓

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- Induction step says $q \to^{xa} q'$ same as $q \to^{x} p \to^{a} q'$
- $\hat{\delta}$ and δ agree on length 1 strings:
  $\hat{\delta}(q, a) = \hat{\delta}(q, \epsilon.a) = \delta(\hat{\delta}(q, \epsilon), a)$ [Inductive definition]
  $= \delta(q, a)$ [Base case of definition]
  x accepted if $\hat{\delta}(s, x) \in F$, otherwise not.

Language accepted by DFA M

$L(M) = \{x | \hat{\delta}(s, x) \in F\}$
Such a set is called a *regular set*. A is a regular set if $A = L(M)$ for a DFA M.
Eg: Is the set $\{x \in \{a, b\}^* | x \text{ contains two consecutive a's}\}$ a regular set?

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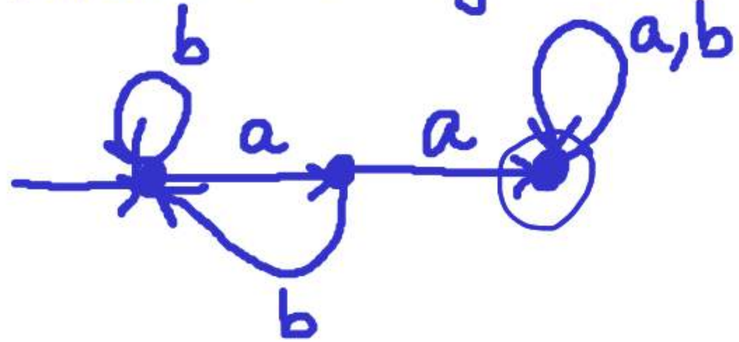
Transition diagram:



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- Similarly, by definition of $\hat{\delta}$, RHS $\hat{\delta}(\hat{\delta}(q, x), \epsilon) = \hat{\delta}(q, x)$
  So $LHS = RHS$.
- Base Case 2: $y = a, a \in \Sigma$: For any x
  $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$ [Inductive definition]
  $= \hat{\delta}(\hat{\delta}(q, x), a)$ [By equality on length 1 strings]

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- $= \hat{\delta}(q', y'a)$ [Base Case 2: length 1 string]

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- $= \hat{\delta}(\hat{\delta}(q, x), y'a) = \hat{\delta}(\hat{\delta}(q, x), y)$