Collapse $\quad p \approx q \iff \forall z \in \Sigma^* \left[\hat{\delta}(p,z) \in F \iff \hat{\delta}(q,z) \in F \right]$

Quotient construction

$\rightarrow$ equivalence classes treated as single states

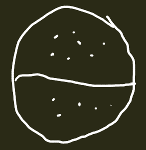
DFA M to an equivalence relation $\equiv_M$ on Σ^*

$$x \equiv_M y \iff \hat{\delta}(s,x) = \hat{\delta}(s,y)$$

MN relation $\equiv$ on Σ^* for a language L

(1) $\quad x \equiv y \Rightarrow \forall a \in \Sigma \, [xa \equiv ya]$ $\quad$ right congruence

(2) $\quad x \equiv y \Rightarrow (x \in L \iff y \in L)$ $\quad$ $\equiv$ refines L



MN' (3) $\quad \equiv$ is of finite index

Construction : Given an MN relation $\equiv$ for L, to construct a DFA $M_{\equiv} = (Q, \Sigma, \delta, s, F)$ such that $\mathcal{L}(M_{\equiv}) = L$.

$Q = \{ [x] \mid x \in \Sigma^* \}$ $\rightarrow$ finite by (3)

$s = [\epsilon]$

$F = \{ [x] \mid x \in L \}$ $\rightarrow$ well-defined by (2)

$\delta([x], a) = [xa]$ $\rightarrow$ well-defined by (1)

$\boxed{\hat{\delta}([x], y) = [xy]}$

$x \in \mathcal{L}(M_{\equiv}) \Leftrightarrow \hat{\delta}(s, x) \in F \Leftrightarrow \hat{\delta}([\epsilon], x) \in F \Leftrightarrow [x] \in F$

$\Leftrightarrow x \in L$

There two constructions are inverses of one another

$$\equiv \;\longmapsto\; M_{\equiv} \;\longmapsto\; \equiv_{M_{\equiv}}$$

$$x \equiv_{M_{\equiv}} y \iff \hat{\delta}([\epsilon], x) = \hat{\delta}([\epsilon], y)$$

$$\iff [x] = [y]$$

$$\iff x \equiv y$$

$$f : Q' \to Q$$

$$f([x]) = \hat{\delta}(s, x)$$

$$M \;\longmapsto\; \equiv_M \;\longmapsto\; M' = M_{\equiv_M}$$

$$M = (Q, \Sigma, \delta, s, F)$$

$$M' = (Q', \Sigma, \delta', s', F')$$

$M = (Q_M, \Sigma, \delta_M, s_M, F_M)$

$N = (Q_N, \Sigma, \delta_N, s_N, F_N)$

M and N are called isomorphic if there exists a $\underset{\sim}{\underline{\text{bijective}}}$ map $f : Q_M \to Q_N$ such that

(1) $f(s_M) = s_N$

(2) $f(F_M) = F_N$

(3) $f(\delta_M(p, a)) = \delta_N(f(p), a)$

$\forall p \in Q_M$ and $\forall a \in \Sigma$

Def: Let L be any language (not necessarily regular).

Define an equivalence relation $\equiv_L$ as follows.

$$x \equiv_L y \iff \forall z \in \Sigma^* \left[xz \in L \iff yz \in L \right]$$

Theorem: $\equiv_L$ is an MN' relation for L.

Proof: $\equiv_L$ refines L. Take $z = \epsilon$ in the defn.

$$x \equiv_L y \Rightarrow \left[x \in L \iff y \in L \right]$$

$\equiv_L$ satisfies right congruence $\quad z = aw$

$$x \equiv_L y \Rightarrow \forall a \in \Sigma \; \forall w \in \Sigma^* \left[\underline{xa}w \in L \iff \underline{ya}w \in L \right]$$

$$\Rightarrow \forall a \in \Sigma \left[xa \equiv_L ya \right]$$

Does such a relation exist for L ?

Yes. Equality relation. $\rightarrow$ The finest MN' relation for L.

We are interested in the coarsest MN' relation for L.

with as few equiv classes as possible

Theorem : $\equiv_L$ is the coarsest MN' relation for L.

Proof : Let $\equiv$ be any MN' relation for L.

TST : $x \equiv y \Rightarrow x \equiv_L y$

$x \equiv y \Rightarrow \forall z \in \Sigma^* [xz \equiv yz]$

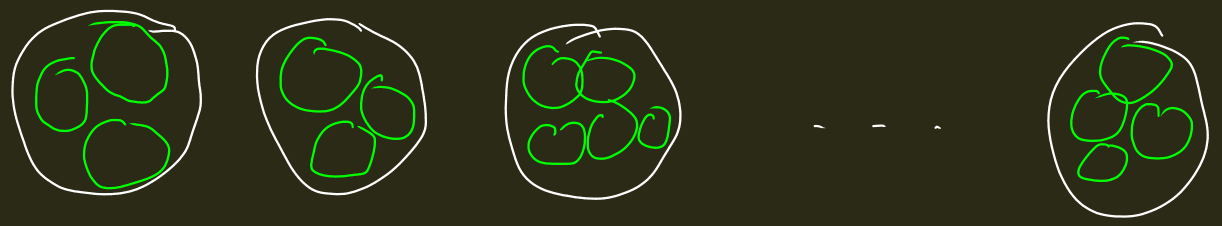
Generalize right congruence for $\equiv$

$\Rightarrow \forall z \in \Sigma^* [xz \in L \Leftrightarrow yz \in L]$

By refinement property

$\boxed{\equiv \subseteq \equiv_L}$

$\Rightarrow x \equiv_L y$



Equivalence classes of $\equiv_L$

Equivalence classes of $\equiv$

Myhill–Nerode theorem

Let $L \subseteq \Sigma^*$. Then the following are equivalent

(1) L is regular

(2) L has an MN relation

(3) $\equiv_L$ is of finite index.

$(1) \Rightarrow (2)$ $\quad L = \mathcal{L}(M)$ $\quad$ Consider $\equiv_M$

$(2) \Rightarrow (3)$ $\quad$ Let $\equiv$ be an MN relation for L

$\equiv \subseteq \equiv_L$ $\quad$ $\equiv_L$ is not finer than $\equiv$

$(3) \Rightarrow (1)$ $\quad$ Do the construction $\equiv_L \longmapsto M_{\equiv_L}$

Application 1

$M = (Q, \Sigma, \delta, s, F)$ be a collapsed DFA without unreachable states

$L = \mathcal{L}(M)$

Then $\equiv_M \; = \; \equiv_L$

Proof: $x \equiv_L y \iff \forall z \in \Sigma^* \left[xz \in L \Leftrightarrow yz \in L \right]$

$\iff \forall z \in \Sigma^* \left[\hat{\delta}(s, xz) \in F \Leftrightarrow \hat{\delta}(s, yz) \in F \right]$

$\iff \forall z \in \Sigma^* \left[\hat{\delta}(\hat{\delta}(s,x), z) \in F \Leftrightarrow \hat{\delta}(\hat{\delta}(s,y), z) \in F \right]$

$\iff \hat{\delta}(s,x) \approx \hat{\delta}(s,y) \iff \hat{\delta}(s,x) = \hat{\delta}(s,y) \iff x \equiv_M y$

Application 2 $\{a^n b^n \mid n \geq 0\}$ is not regular.

$i \neq j$ $\quad a^i b^i \in L \qquad a^j b^i \notin L$

$a^i b^j \notin L \qquad a^j b^j \in L$

$$a^i \not\equiv_L a^j$$

$[a^0], [a^1], [a^2], [a^3], \ldots$ are all distinct

$\equiv_L$ is not of finite index

L is not regular.