An algorithm for the Quotient Construction

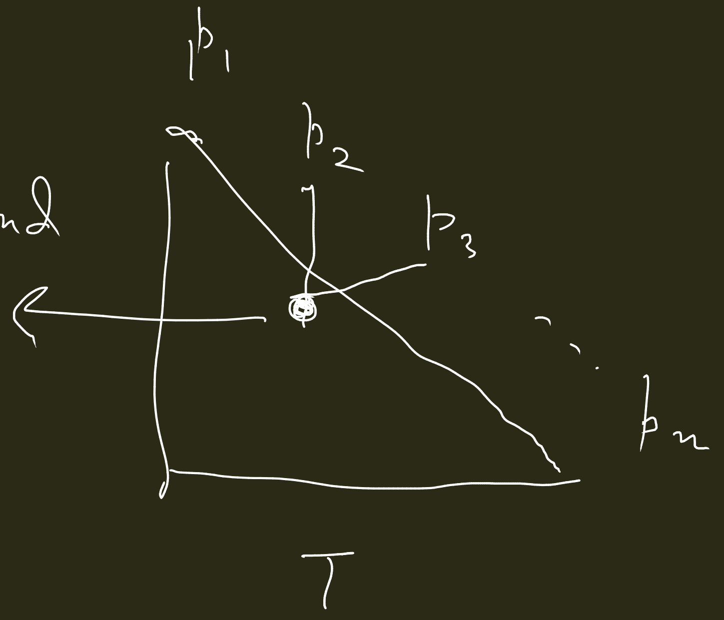
- Locates non-equivalent pairs
- maintains a 2-D array indexed by sets

Mark (p, q)-th entry
if $p \in F$ and $q \notin F$
or $p \notin F$ and $q \in F$

init

- if not found
non-eqvt

X if found
non-eqvt
marking

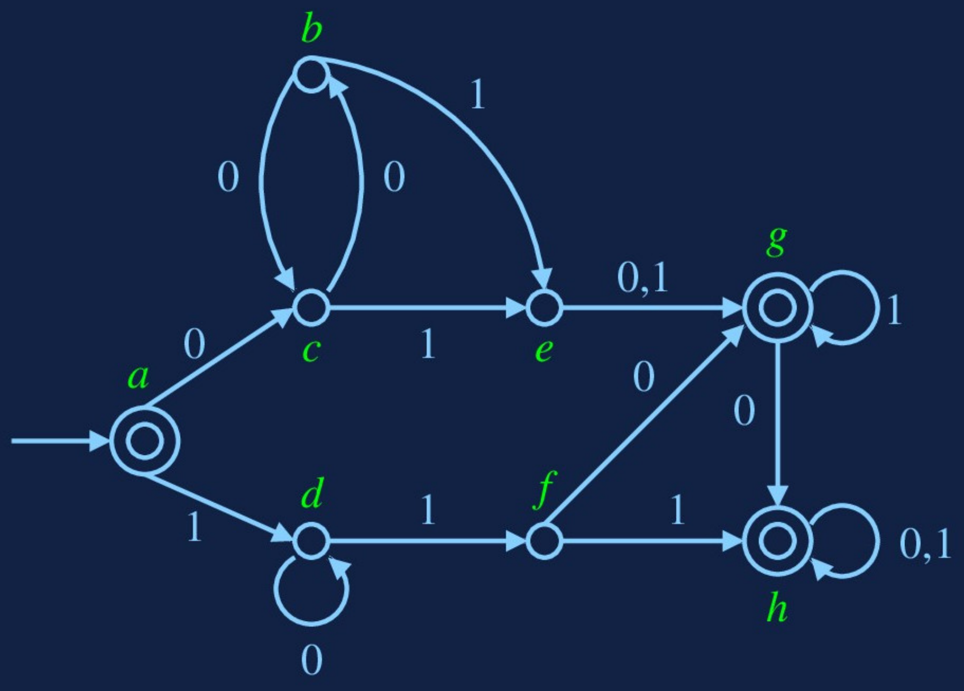


repeat until no further marking is possible

loop

find p and q and $a \in \Sigma$ such that $T(\delta(p, a), \delta(q, a))$ is marked.

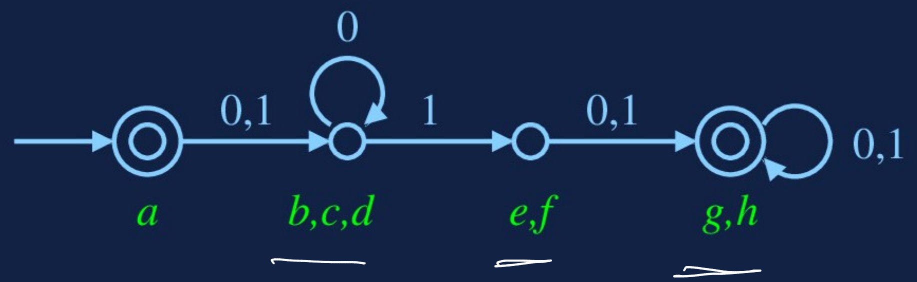
If found, mark $T(p, q)$.

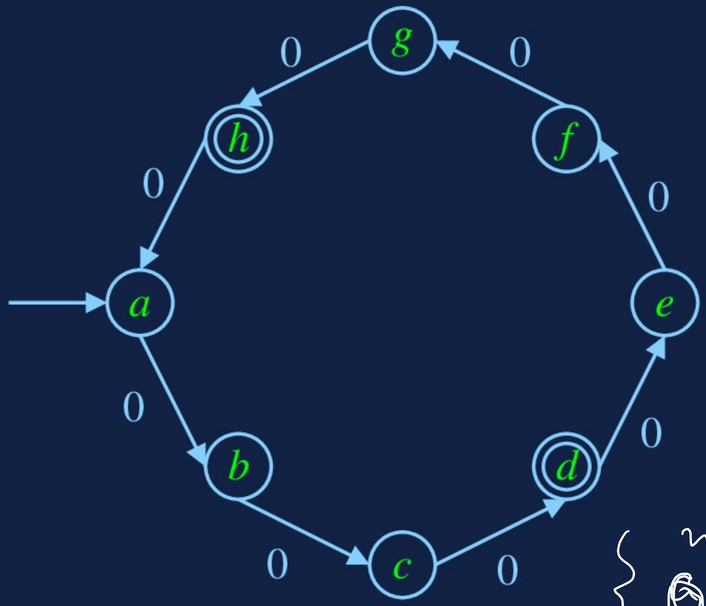


	0	1
a	c	d
b	c	e
c	b	e
d	d	f
e	g	g
f	g	h
g	h	g
h	h	h

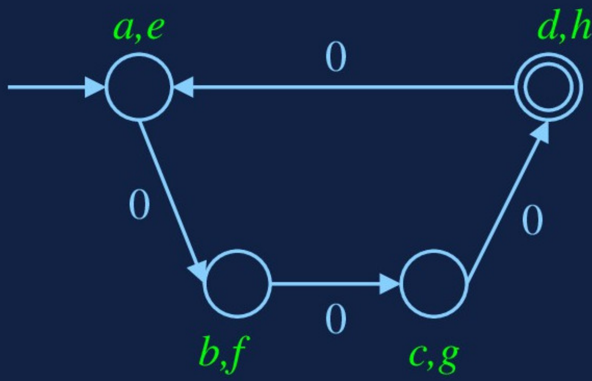
a							
×	b						
×	–	c					
×	–	–	d				
×	–	–	–	e			
×	–	–	–	–	f		
–	×	×	×	×	×	g	
–	×	×	×	×	×	–	h

a							
×	b						
×	–	c					
×	–	–	d				
×	×	×	×	e			
×	×	×	×	–	f		
×	×	×	×	×	×	g	
×	×	×	×	×	×	–	h





$\{ 0^n \mid n \equiv 3 \pmod 4 \}$



$\{ 0^n \mid n \equiv 3 \text{ or } 7 \pmod 8 \}$

a							
–	b						
–	–	c					
0	0	0	d				
–	–	–	0	e			
–	–	–	0	–	f		
–	–	–	0	–	–	g	
0	0	0	–	0	0	0	h

a							
–	b						
1	1	c					
0	0	0	d				
–	–	1	0	e			
–	–	1	0	–	f		
1	1	–	0	1	1	g	
0	0	0	–	0	0	0	h

a							
2	b						
1	1	c					
0	0	0	d				
–	2	1	0	e			
2	–	1	0	2	f		
1	1	–	0	1	1	g	
0	0	0	–	0	0	0	h

__Running time__ $\quad T$ has $\binom{n}{2} = O(n^2)$ entries

$n = |Q| \qquad |\Sigma| = \text{constant}$

Init : $O(n^2)$

Each iteration : $O(n^2)$

Max no of iteration is $O(n^2)$

$\Bigg] \; O(n^4)$ poly-time

__Correctness__ $\quad p$ and q are marked

$\Longleftrightarrow$ p and q are not equivalent.

$\Rightarrow$ Evident $\qquad \Leftarrow \quad \hat{\delta}(p, z) \in F$ and $\hat{\delta}(q, z) \notin F$ for some $z \in \Sigma^*$

Proceed by induction on $|z|$.

Partition of Σ* produced by a DFA

$M = (Q, \Sigma, \delta, s, F)$ DFA

$\hookrightarrow$ no unreachable states

$L = \mathcal{L}(M)$

$x \in \Sigma^*$ $\quad \hat{\delta}(s, x) \longrightarrow$ a unique element of Q

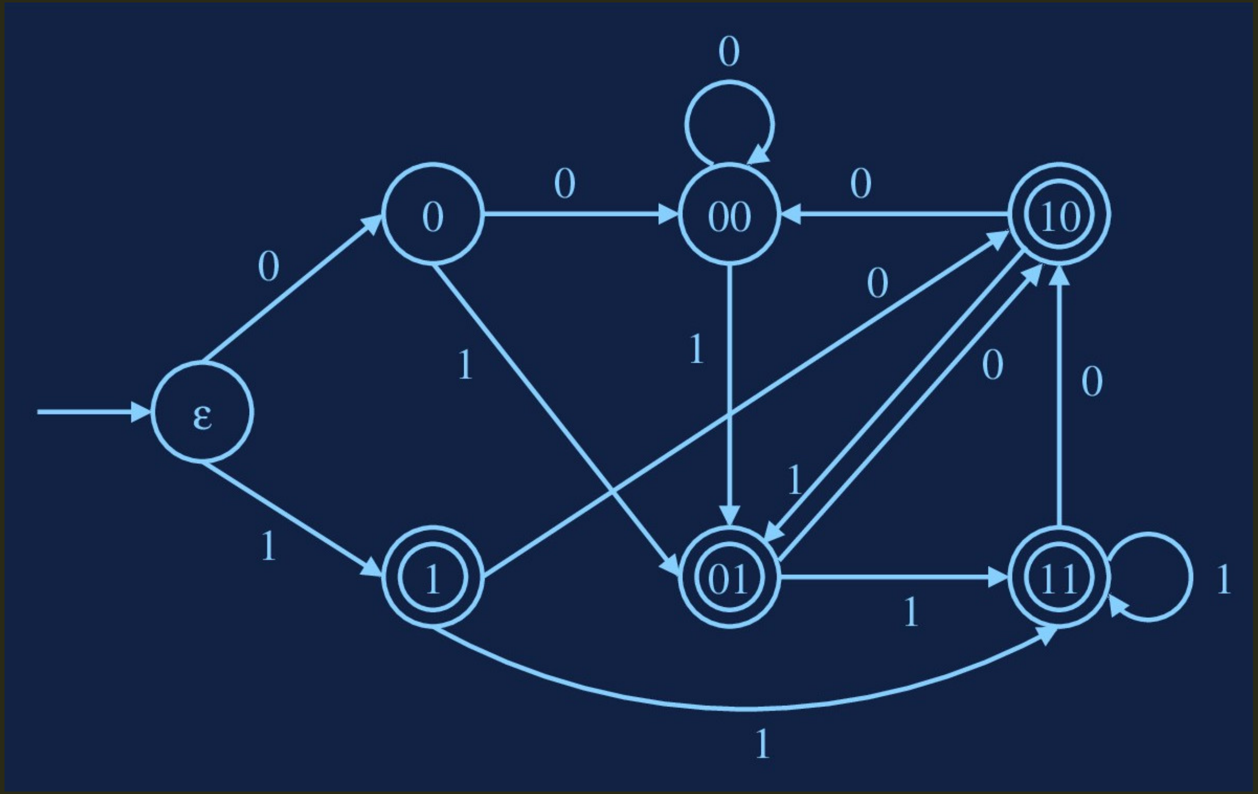
$$\emptyset \neq L_q = \{ x \in \Sigma^* \mid \hat{\delta}(s, x) = q \} \subseteq \Sigma^*$$

L_q, $q \in Q$, form a partition of Σ^*

L is the disjoint union of L_f, $f \in F$.

$$p \neq q \Rightarrow L_p \cap L_q = \emptyset \qquad \sim L = \bigcup_{q \in Q - F} L_q$$

$L = \{x \in \{0,1\}^* \mid x$ contains at least one 1 in its last two positions$\}$



$L_\varepsilon = \{ \varepsilon \}$

$L_0 = \{ 0 \}$

$L_1 = \{ 1 \}$

$L_{00} = \mathscr{L}((0 + 1)^*00)$

$L_{01} = \mathscr{L}((0 + 1)^*01)$

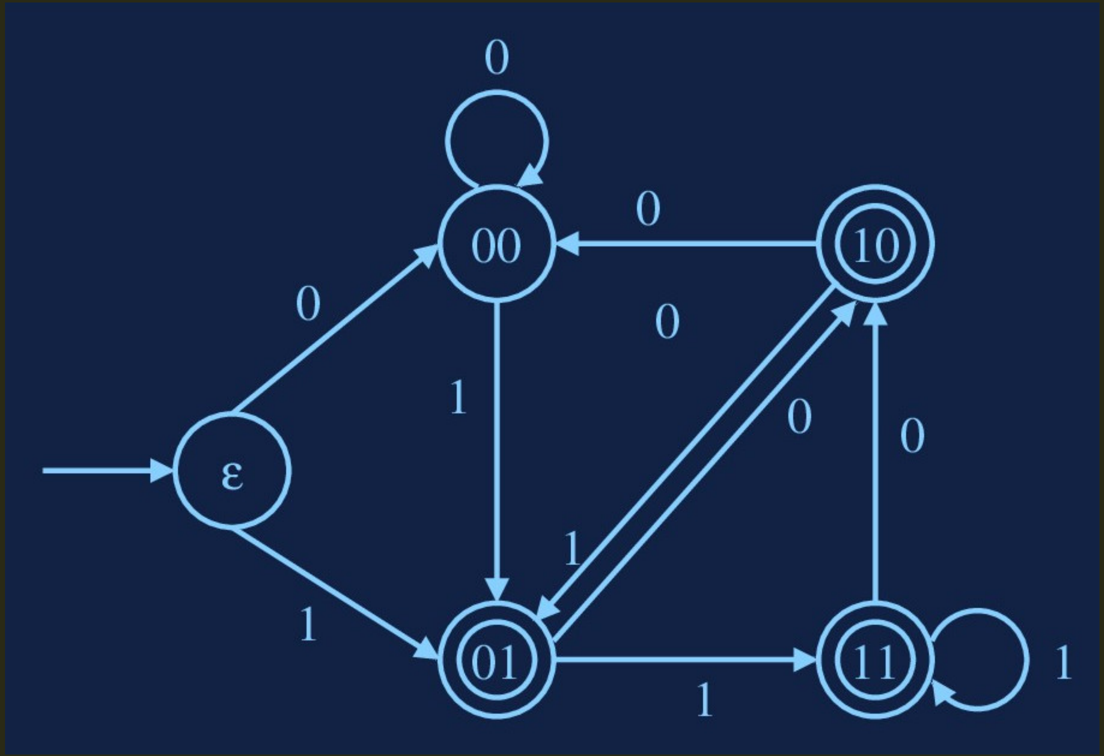
$L_{10} = \mathscr{L}((0 + 1)^*10)$

$L_{11} = \mathscr{L}((0 + 1)^*11)$

$L = L_1 \cup L_{01} \cup L_{10} \cup L_{11}$

7 - part partition

$L = \{x \in \{0,1\}^* \mid x \text{ contains at least one 1 in its last two positions}\}$



$L_\varepsilon = \{ \varepsilon \}$

$L_{00} = \{ 0 \} \cup \mathscr{L} ((0 + 1)^*00)$

$L_{01} = \{ 1 \} \cup \mathscr{L} ((0 + 1)^*01)$

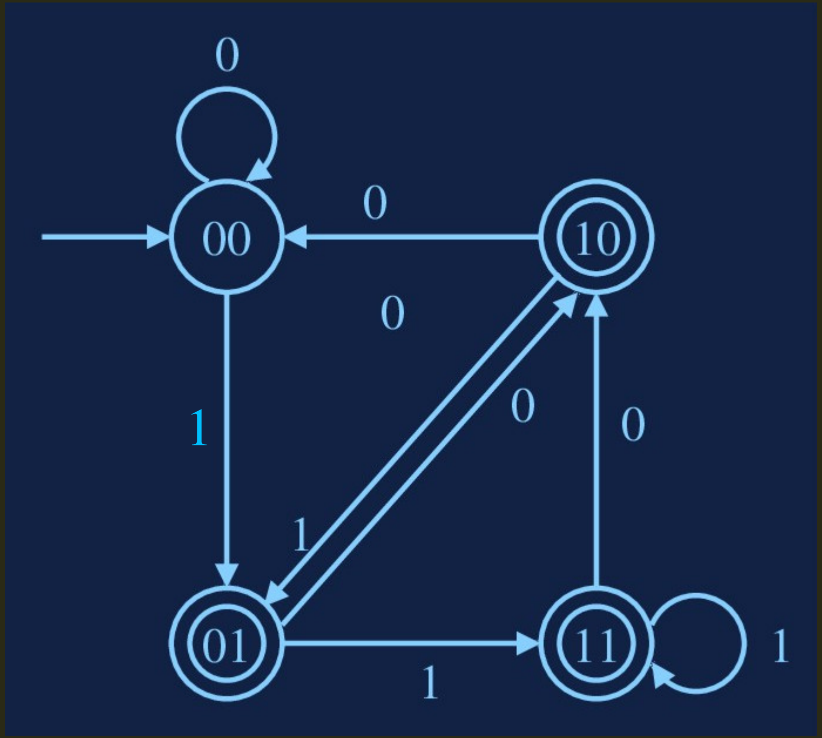
$L_{10} = \mathscr{L} ((0 + 1)^*10)$

$L_{11} = \mathscr{L} ((0 + 1)^*11)$

$L = L_{01} \cup L_{10} \cup L_{11}$

5 parts

$L = \{x \in \{0,1\}^* \mid x$ contains at least one 1 in its last two positions$\}$



$L_{00} = \{\ \varepsilon, 0\ \} \cup \mathscr{L}(\ (0+1)^*00\ )$

$L_{01} = \{\ 1\ \} \cup \mathscr{L}(\ (0+1)^*01\ )$

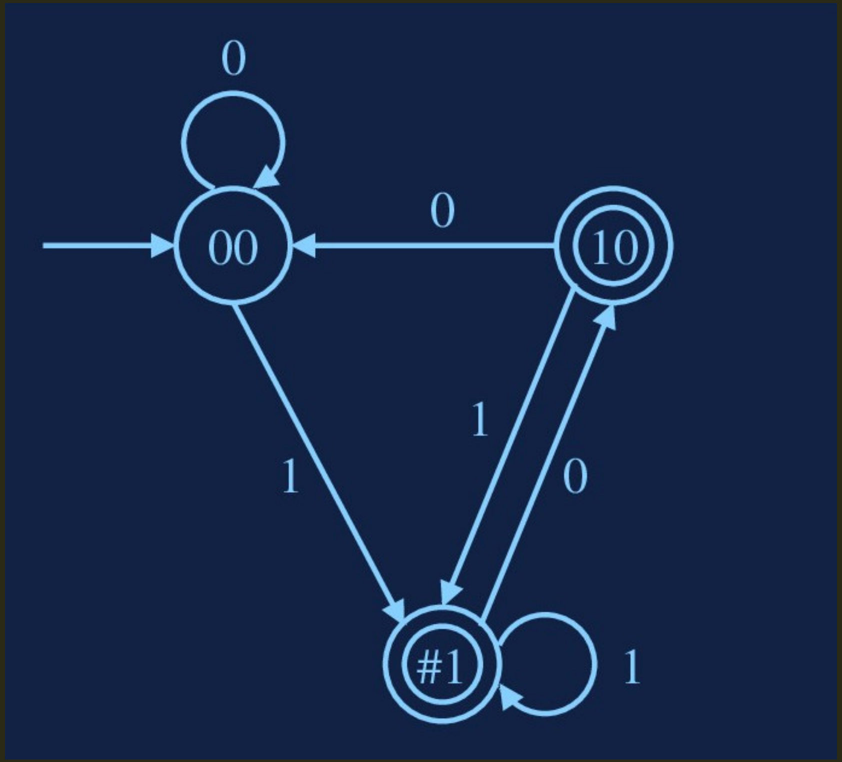
$L_{10} = \mathscr{L}(\ (0+1)^*10\ )$

$L_{11} = \mathscr{L}(\ (0+1)^*11\ )$

$L = L_{01} \cup L_{10} \cup L_{11}$

4 parts

$L = \{x \in \{0,1\}^* \mid x$ contains at least one 1 in its last two positions$\}$



As the # of states decreases, the partition becomes coarser.

$$L_{00} = \{ \varepsilon, 0 \} \cup \mathscr{L}((0+1)^*00)$$

$$L_{10} = \mathscr{L}((0+1)^*10)$$

$$L_{\#1} = \mathscr{L}((0+1)^*1)$$

$$L = L_{10} \cup L_{\#1}$$

3 parts

Not all partitions are realizable by DFA

L = {x ∈ {0,1}* | x contains at least one 1 in its last two positions}

Can I have a DFA giving a 2-part partition?

— No.

r $\quad$ t

$$\hat{\delta}(s, 01)$$
$$= \hat{\delta}(s, 10)$$
$$= t$$

$$\delta(t, 0) \stackrel{?}{=} r \text{ or } t \, ?$$

Coarsest possible partition

≡ the minimal DFA

Myhill–Nerode Relations

Partitions $\equiv$ Equivalence relations

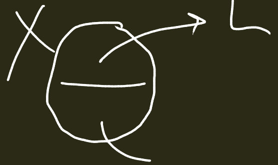
Given M, define an equivalence relation $\equiv_M$ on Σ^* as

MN relation for L

$$x \equiv_M y \quad \text{if and only if} \quad \hat{\delta}(s, x) = \hat{\delta}(s, y)$$

- Left congruence : $x \equiv_M y \Rightarrow xa \equiv_M ya \ \forall a \in \Sigma$

- $\equiv_M$ refines $L = \mathcal{L}(M)$



L is the disjoint union of some equivalence classes

$x \equiv_M y \Rightarrow (x \in L \Longleftrightarrow y \in L)$ $\sim L$

- $\equiv_M$ has finite index $\longrightarrow$ finitely many equiv classes.