Some other ways to prove non-regularity

- Using closure properties

$$L_4 = \{ w \in \{a, b\}^* \mid \#a(w) = \#b(w) \}$$

abba, babaabb $\in L_4$

Suppose that L_4 is regular.

$\mathcal{L}(a^* b^*)$ is regular.

$\Rightarrow L_4 \cap \mathcal{L}(a^* b^*)$ is also regular

$= \{ a^n b^n \mid n \geq 0 \} = L_1$

L language

$L^R = \text{rev } L$

$= \{ x \mid x^R \in L \}$

Reg languages are closed under reversal.

$$L_5 = \{ a^m b^n \mid m, n \geq 0, \; m \leq n \}$$

Suppose L_5 is regular. rev L_5 is regular.

$$\text{rev } L_5 = \{ b^n a^m \mid m, n \geq 0, \ m \leq n \}$$

$$= \{ b^m a^n \mid m, n \geq 0, \ m \geq n \}$$

is regular

$$\Rightarrow \quad L_5' = \{ a^m b^n \mid m, n \geq 0, \ m \geq n \}$$ is also regular

$L_5 \cap L_5'$ is also regular

$$= \{ a^n b^n \mid n \geq 0 \} = L_1 \ ↯$$

Ultimate periodicity

$$S \subseteq \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$$

S is called ultimately (eventually) periodic if there exist integer constants $n_0 \in \mathbb{N}_0$ and $p \in \mathbb{N}$ such that

$$\forall n \geq n_0 \left[n \in S \overset{\Rightarrow ✓}{\underset{\Leftarrow ?}{\Longleftrightarrow}} n + p \in S \right]$$

a period

$$S = \{2, 3, 5, 7\} \cup \{10, 12, 14, 16, 18, \ldots\}$$
$$\cup \{12, 15, 18, 21, \ldots\}$$

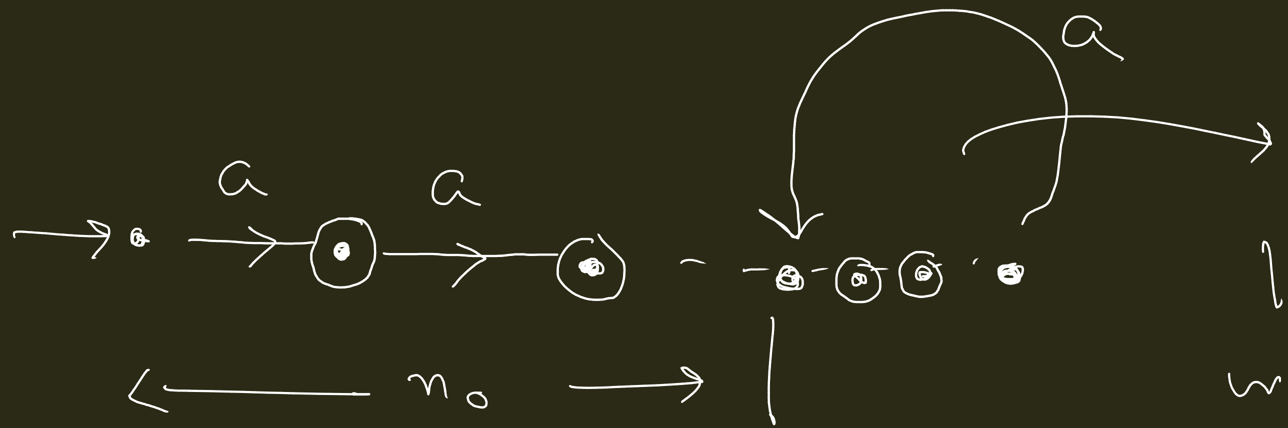
$n_0 = 10, \quad p = 6$

$L \subseteq \Sigma^*$ $\quad$ $\text{lengths}(L) = \{ |x| \mid x \in L \} \subseteq \mathbb{N}_0$

<u>Theorem</u> Let L be a language over a singleton alphabet $\Sigma = \{a\}$. Then

L is regular $\iff$ $\text{lengths}(L)$ is u.p.

<u>Proof</u>: $\Rightarrow$ L is regular. $L = \mathcal{L}(D)$ for some DFA D.



length p

$\text{lengths}(L)$ is u.p. with period p

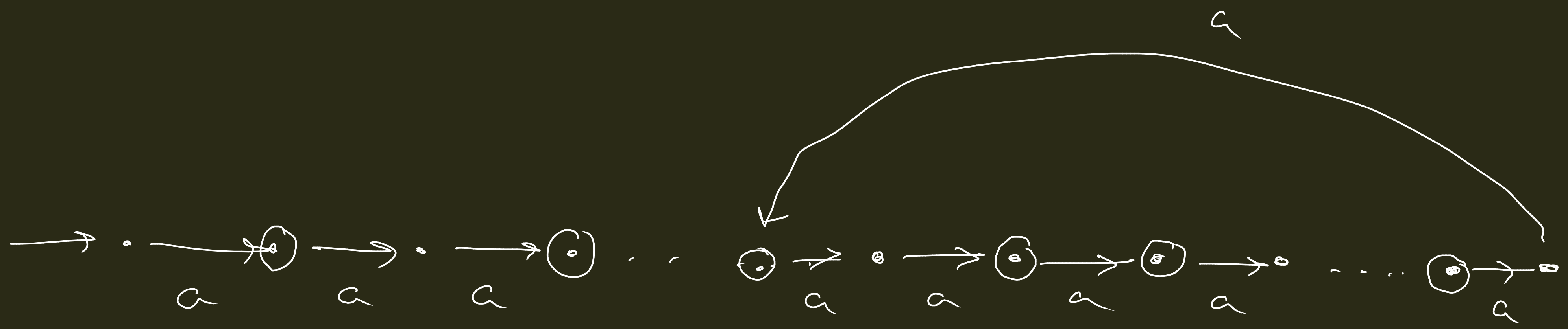
$\Longleftarrow$ $\text{lengths}(L)$ is u.p.

$$\underset{b_1\, b_2 \ldots b_l}{\{\ \ldots\ \}} \cup \{a_0, a_0+p, a_0+2p, \ldots\}$$

$$\cup \{a_1, a_1+p, a_1+2p, \ldots\}$$

$$\ldots$$

$$\cup \{a_k, a_k+p, a_k+2p, \ldots\}$$



$L_5 = \{ a^{n^2} \mid n \geq 0 \}$ and $L_6 = \{ a^{n!} \mid n \geq 0 \}$

are not regular.

<u>Theorem</u>: Let Σ be an arbitrary alphabet

L is regular $\Rightarrow$ lengths(L) is u.p.

<u>Pf</u>: Let k be a PLC.

Take $n_0 = k$, $p = \mathrm{lcm}(1, 2, \ldots, k)$.

[Flaw]

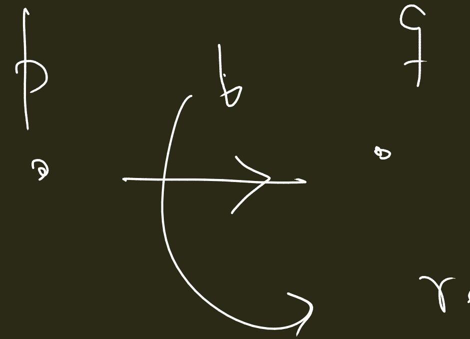
(not necessarily the smallest)

L is regular over $\Sigma = \{a, b, c, \dots\}$

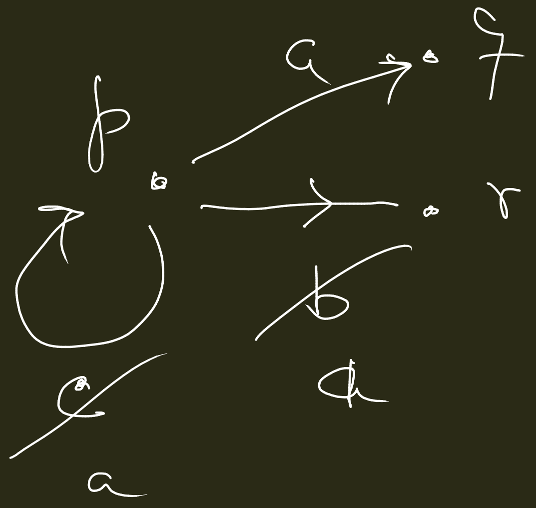
$L = \mathcal{L}(D)$, D a DFA

$\downarrow$ $\qquad L' = \mathcal{L}(D')$.

D' an NFA over $\{a\}$



replace by a



L' is regular

$\text{lengths}(L) = \text{lengths}(L')$.

The converse is not necessarily true.

$L_1 = \{a^n b^n \mid n \geq 0\}$ is not regular

But $\text{lengths}(L_1) = \{0, 2, 4, 6, 8, \ldots\}$ is u.p.

Application

$L_7 = \{a^n b^{n^2} \mid n \geq 0\}$

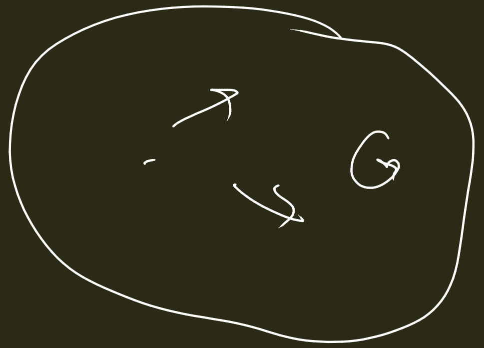
is not regular

$L_8 = \{w \in \{a, b\}^* \mid \#b(w) = (\#a(w))^2\}$

is not regular.

State minimization

- Design an equivalent DFA with as few states as possible
  * Remove all unreachable states (DFS/BFS)
  * Collapse equivalent states



D DFA

- $\mathcal{L}(D) = \mathcal{L}(D')$

  $\tilde{D} \stackrel{?}{=} \tilde{D}'$ ✓

$D \to$ DFA over Σ accepting $L = \mathcal{L}(D)$

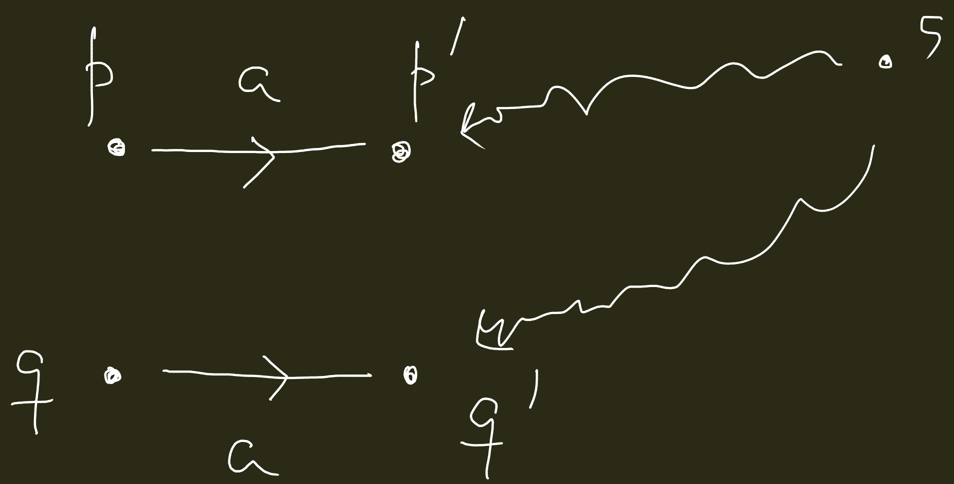
$D = (Q, \Sigma, \delta, s, F)$

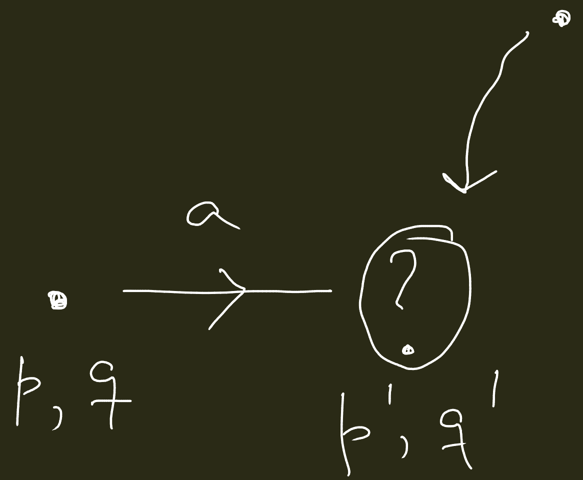
$p, q \in Q$ $\quad$ p is equivalent to q

$$p \approx q$$

$$\Leftrightarrow \forall x \in \Sigma^* \left[\hat{\delta}(p, x) \in F \Leftrightarrow \hat{\delta}(q, x) \in F \right]$$

Aim: To merge p and q to a single state.





$p \approx q$

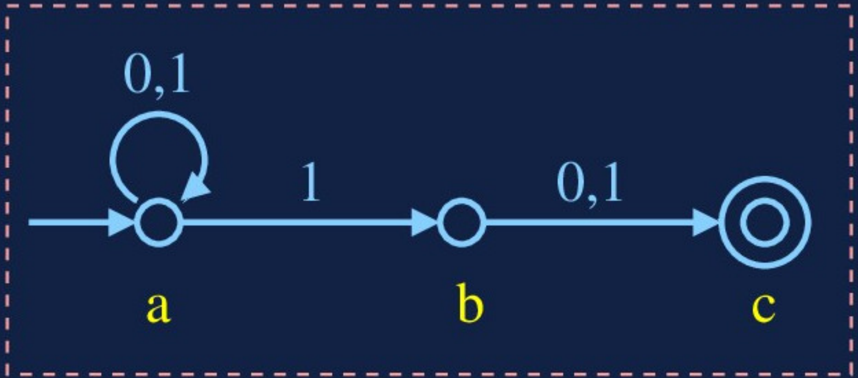
if $p' = q'$, no problem

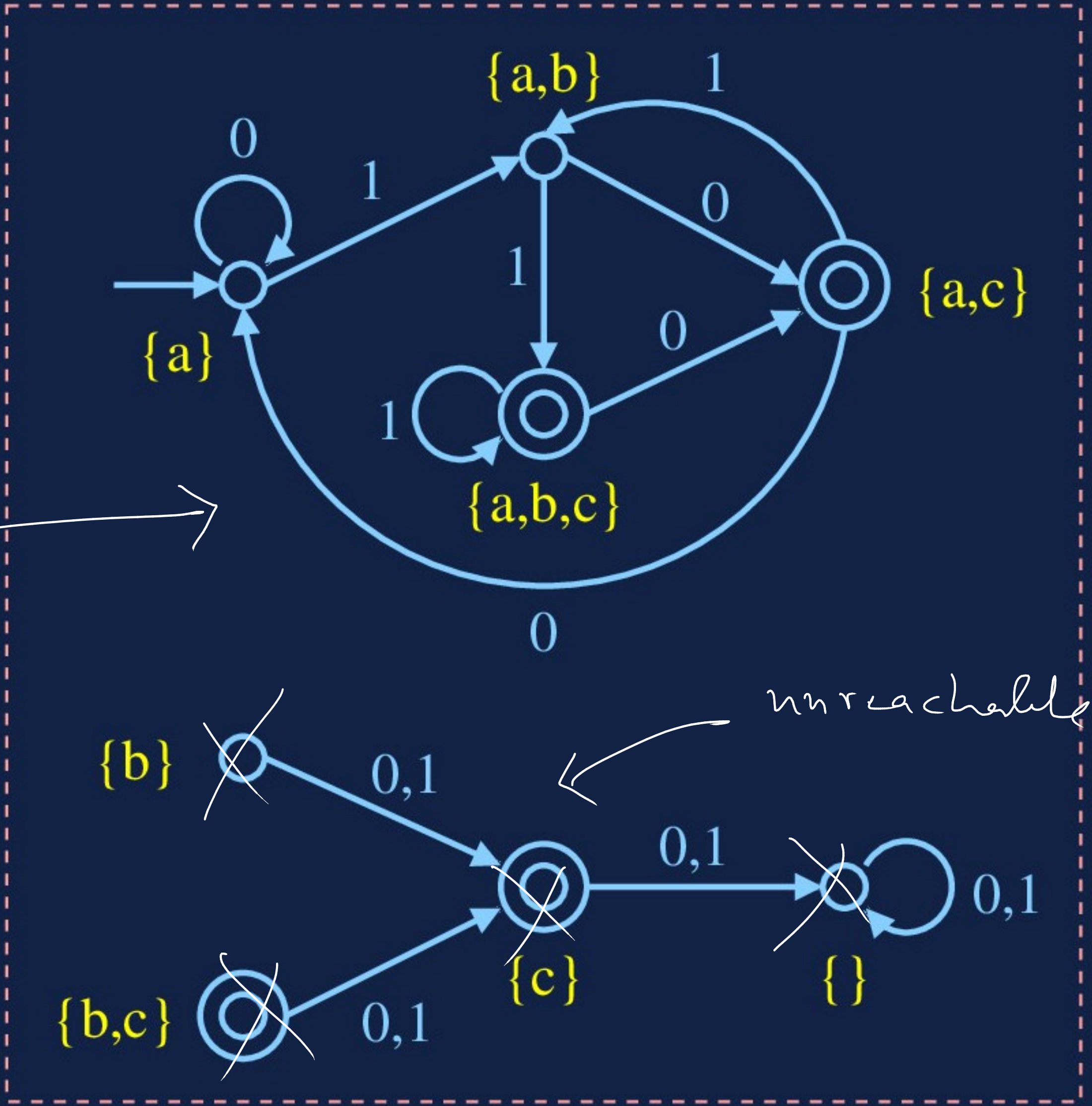
if $p' \neq q'$, then $p' \approx q'$

$x = ay$

$\hat{\delta}(p, x) \in F \not\Leftrightarrow \hat{\delta}(q, x) \in F$

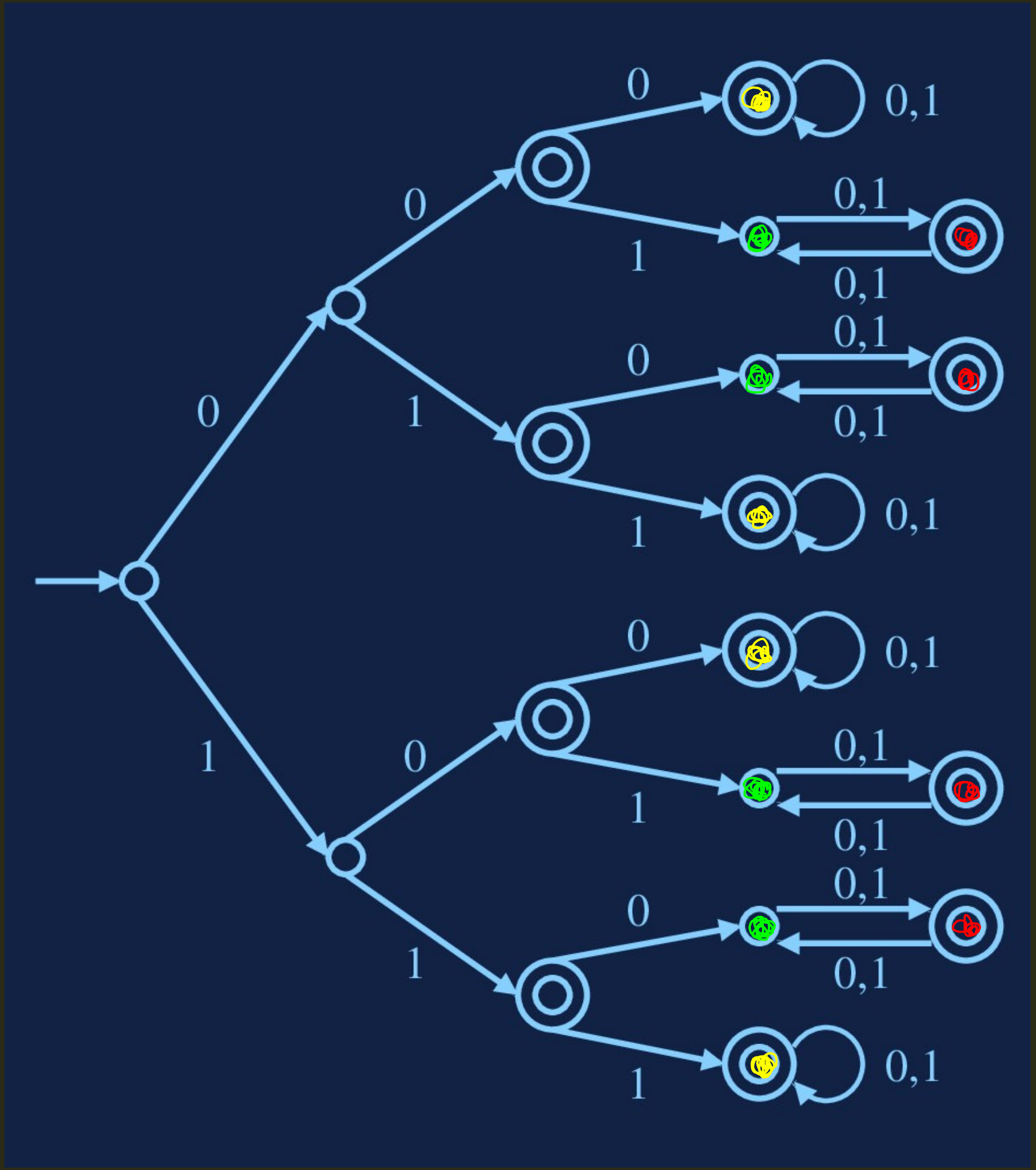
$\hat{\delta}(p', y) \in F \not\Leftrightarrow \hat{\delta}(q', y) \in F$





a minimal DFA

the

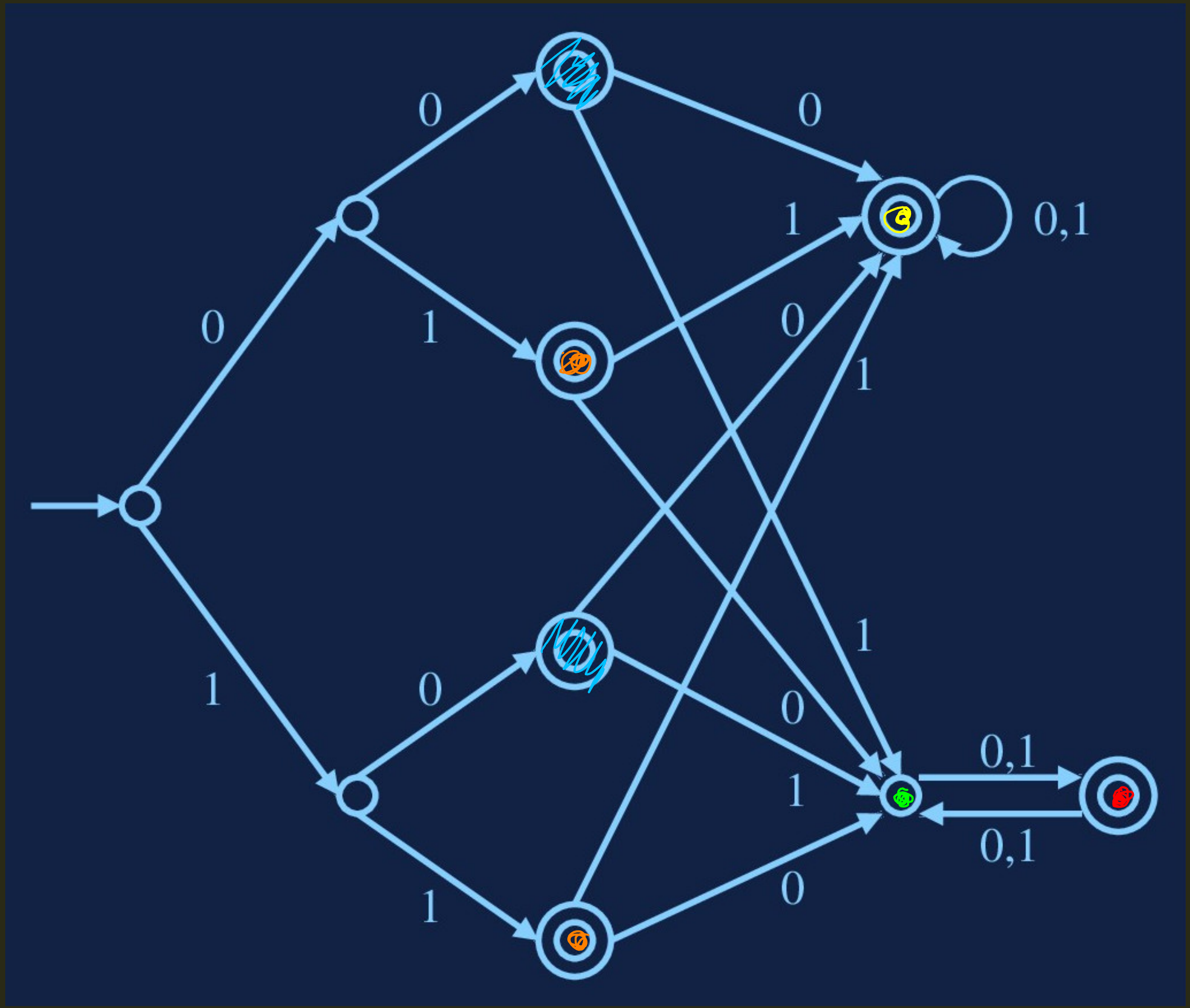


$L = \{ w \mid |w| \text{ is even} \} \cup \{ w \mid \text{second and third symbols of } w \text{ are the same} \}$

Collapse the final states marked yellow

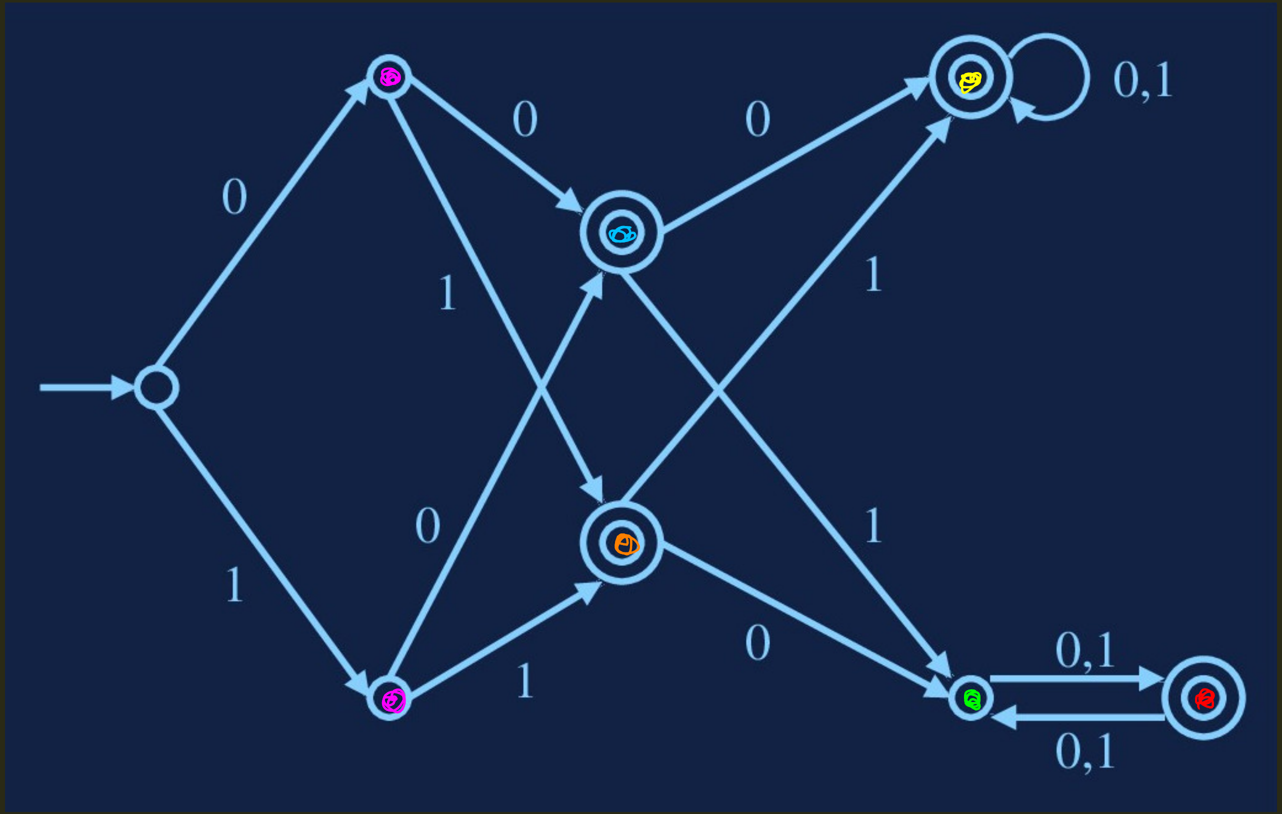
Collapse green states

Collapse red states

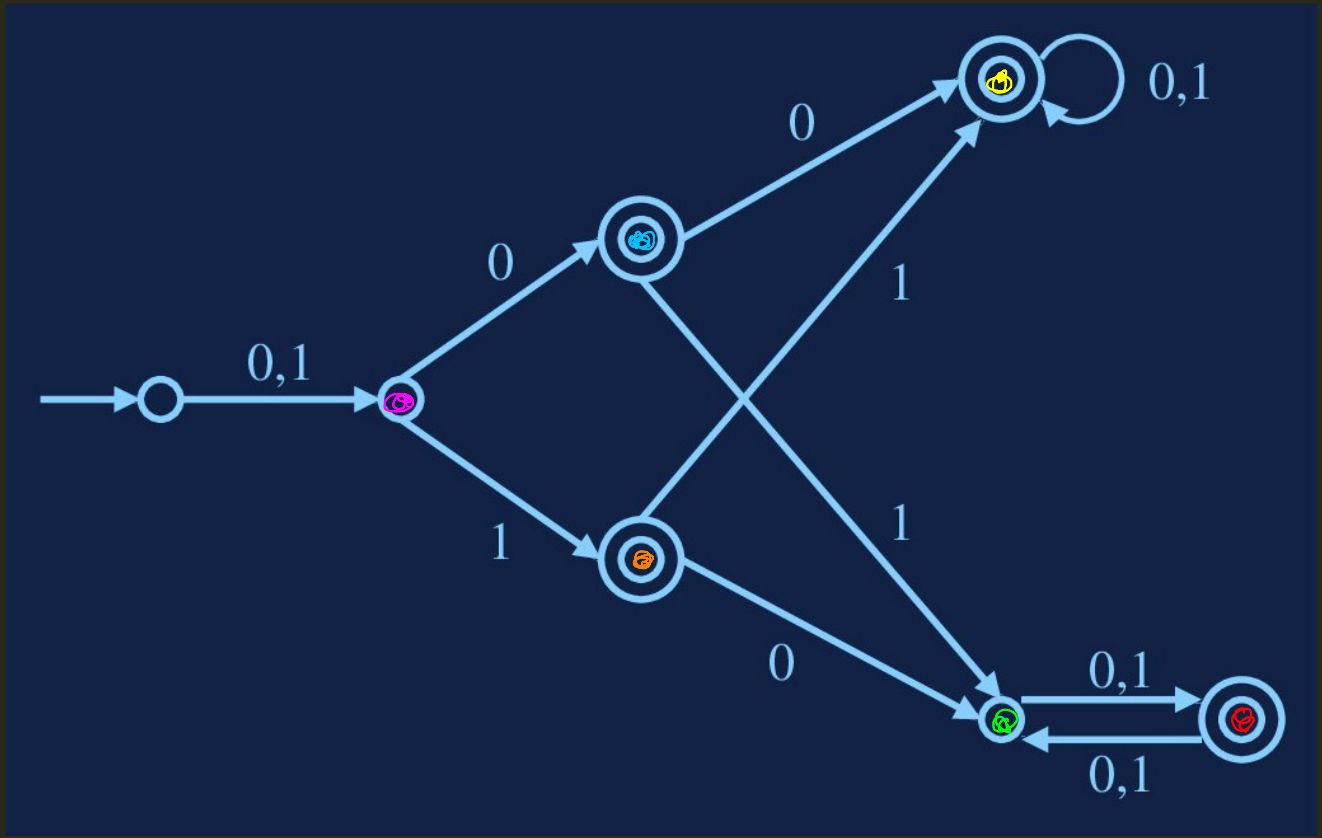


Collapse blue states

Collapse amber states



pink states can also be collapsed



No further
collapsing
possibility

$D = (Q, \Sigma, \delta, s, F)$ Define $\approx$ on Q as before.

Claim: $\approx$ is an equivalence relation on Q.

Proof $p \approx p$

$p \approx q \Rightarrow q \approx p$

$p \approx q$ and $q \approx r \Rightarrow p \approx r$

Quotient construction

$D' = (Q', \Sigma, \delta', s', F')$

$Q' = Q/\approx = \{ [q] \mid q \in Q \}$

$s' = [s]$

$F' = \{ [f] \mid f \in F \}$

safe

well-defined

$\delta'([p], a) = [\delta(p, a)]$

<u>Claim</u> : $\hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$

for all $x \in \Sigma^*$.

<u>Pf</u> : Induction on $|x|$.

<u>Claim</u> : $[p] \in F' \iff p \in F$.

$$
\begin{aligned}
x \in \mathcal{L}(D) &\iff \hat{\delta}(s, x) \in F \\
&\iff [\hat{\delta}(s, x)] \in F' \\
&\iff \hat{\delta}'([s], x) \in F' \\
&\iff x \in \mathcal{L}(D')
\end{aligned}
$$

<u>$\mathcal{L}(D) = \mathcal{L}(D')$</u>