Pumping lemma

Let L be a regular language. Then, there exists a positive constant k such that given any string $uvw \in L$ with $|v| \geq k$ we have the decomposition $v = xyz$ satisfying

(1) y is non-empty

(2) $|y| \leq k$

(3) $uxy^i zw \in L$ for all $i \geq 0$.

$w = a^{\lceil k/2 \rceil} b^{\lceil k/2 \rceil}$ — 3 cases

$w = \underbrace{a^k}_{v} \underbrace{b^k}_{w}$ — 1 case

$u = \epsilon$

introduction of u and w forces pump in / out in v

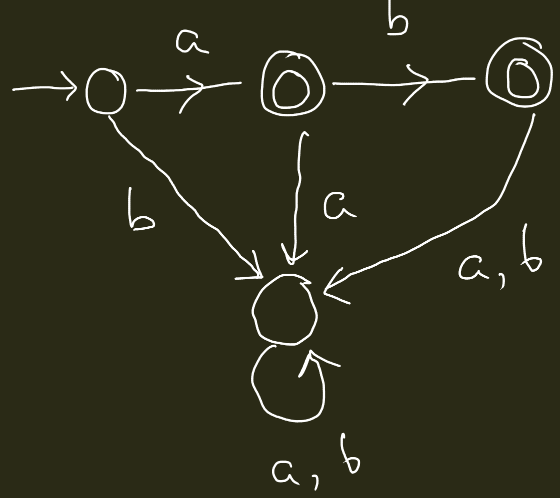
may $\Rightarrow$ proof simplification

Note : While using the lemma, forget about the DFA.

Game with the demon

Demon	You
	1. Assume L is regular.
2. Declares k	
	3. Prepare $uvw \in L$ with $\lvert v \rvert \geq k$.
4. $v = xyz$, $1 \leq \lvert y \rvert \leq k$, $uxy^i zw \in L$ for all $i \geq 0$	
	5. Choose i for a contradiction

$L = \{a, ab\}$



\# of states is 4.

Demon gives you $k = 3$.

Example 2 $\quad L_2 = \{ a^{n^2} \mid n \geq 0 \}$

Assume that L_2 is regular. Let k be a PLC for L_2.

Take $u = \epsilon$, $v = a^{k^2}$, $w = \epsilon$ $\quad$ ($k^2 \geq k$ for all $k > 0$)

We have $\quad v = xyz \quad$ with $\quad 1 \leq |y| \leq k$

and $\quad uxy^i zw \in L_2 \quad \forall i \geq 0$.

Take $i = 2 \qquad |y| = l$

$k^2 + l \neq k^2$

$uxy^2zw = a^{k^2 + l} \in L_2 \qquad$ since $l > 0$.

$k^2 + l \leq k^2 + k$

$< k^2 + 2k + 1 = (k+1)^2$.

Example 3

$$L_3 = \{ a^m b^n \mid m, n \geq 0, \quad m \neq n \}$$

Suppose L_3 is reg. Let k be a PLC for L_3.

$$u = \epsilon, \quad v = a^k, \quad w = b^{k+1}$$

$$v = xyz \qquad 1 \leq \underset{= l}{|y|} \leq k \qquad u x y^i z w \in L \quad \forall i \geq 0$$

* No control on y or $|y|$

$i = 2$ $\qquad a^{k+l} b^{k+1} \in L$

Take $l = 1$

* Do not think about the DFA.

$$u = \epsilon, \quad v = a^k, \quad w = b^{k+l}$$

Replay with demon

$$v = x' y' z' \qquad 1 \leq \underset{= l'}{|y'|} \leq k$$

Demon does not guarantee $l' = l$.

$u = \epsilon, \quad \underline{v = a^k}, \quad w = b^{k+k!}$

$v = xyz \qquad |y| = \ell \qquad 1 \leq \ell \leq k$

$uxy^izw \in L \qquad \forall\, i \geq 0$

Take $i = (k!/\ell) + 1$

$\to$ an integer

$uxy^izw = a^{k+k!}\, b^{k+k!} \in L$