To prove regular
- DFA
- NFA, ε-NFA
- patterns
- regular expr
- Linear grammar

finite representations

There must be non-regular languages
- Identify
- Prove non-regularity

Pumping Lemma
(PL)

Proof by contradiction

To prove that L is not regular.

Suppose L is regular. $\Rightarrow$ $L = \mathcal{L}(D)$ $\rightarrow$ DFA

D has k states.

Finite languages are regular.

Since L is infinite, it contains a string w with $|w| \geq k$.

$|w| = n \geq k$

D accepts w.

$w = a_1 a_2 a_3 \cdots a_n$

$$\begin{array}{c} q \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{a_3} p_3 \rightarrow \cdots \xrightarrow{a_n} p_n \in F \\ \| \\ p_0 \end{array}$$

$p_0, p_1, p_2, p_3, \ldots, p_n \rightarrow n+1 \geq k+1$ states

We must have $p_i = p_j$ for $0 \leq \underline{i < j} \leq n$.

$$p_0 \xrightarrow{x} p_i \xrightarrow{y} p_j = p_i \xrightarrow{z} p_n \in F$$

$w_1 = w = \underline{xyz}$ $\quad |y| \geq 1$ $\quad$ pumping out $\rightarrow$ $w_0 = xz \in L$

pumping in $\rightarrow$ $w_2 = xy^2z = xy.yz \in L$

$$\boxed{w_i = xy^iz \in L \quad \text{for all } i \geq 0}$$

By pumping in/out, we will arrive at a w_i which cannot be in L.

Example $L = \{a^n b^n \mid n \geq 0\}$ is not regular.

ϵ, ab, $aabb$, $aaabbb$, ...

Pf: Suppose L is regular. $L = \mathcal{L}(D)$.

$k \rightarrow$ no. of states in D

$w = a^{\lceil k/2 \rceil} b^{\lceil k/2 \rceil}$ $\quad n = \lceil k/2 \rceil + \lceil k/2 \rceil \geq k$

$= xyz$ with $|y| \geq 1$

and $w_i = xy^i z \in L$

Case 1 y is inside the block of a's.

$$\underbrace{aa\ldots a}_{\text{Case 1}}\;\underbrace{bb\ldots b}_{\text{Case 2}}$$

Case 2 y is inside the block of b's

Case 3 y spans across the $a-b$ boundary

$|y| = \ell \geq 1$

Case 1 : $w_0 = xz = a^{\lceil k/2 \rceil - \ell} b^{\lceil k/2 \rceil} \in L$

$w_2 = xy^2z = a^{\lceil k/2 \rceil + \ell} b^{\lceil k/2 \rceil} \in L$

Case 2 : $w_0 = xz = a^{\lceil k/2 \rceil} b^{\lceil k/2 \rceil - \ell} \in L$

Case 3 :

$$y = a^r b^t \qquad aa \ldots \underline{a\,b}\,b \ldots b$$

$$w_0 = xz = a^{\lceil k/2 \rceil - r}\, b^{\lceil k/2 \rceil - t}$$

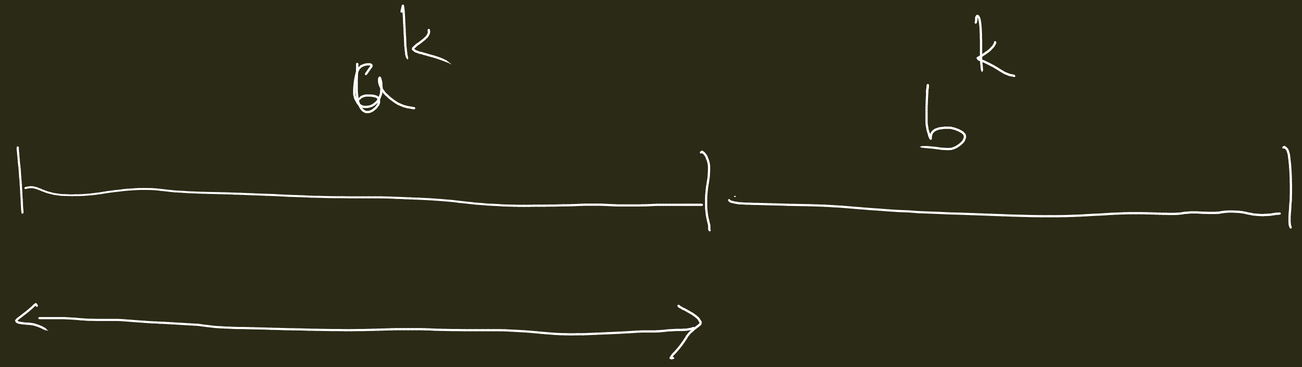
if $r = t$, no contradiction

$$w_2 = xy^2z = a^{\lceil k/2 \rceil - r}\, a^r \underline{b^t a^r} b^t\, b^{\lceil k/2 \rceil - t}$$

$\hookrightarrow$ not of the form $a^* b^*$

Simpler proof

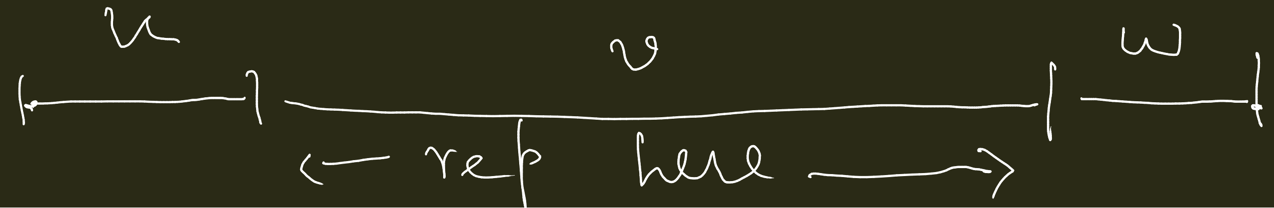
$$w = a^k b^k$$



repetition will happen here

$$y = a^r \text{ for some } r \geq 1.$$

Only case 1 suffices.



$|v| \geq k$