Formal Languages and Automata Theory

A note on ε-NFA

Let $N = (Q, \Sigma, \Delta, S, F)$ be an ε-NFA. Here, Q is a finite set of states, Σ is the input alphabet, $S \subseteq Q$ is the set of start states, $F \subseteq Q$ is the set of final states, and $\Delta : Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$ is the transition function satisfying $q \in \Delta(q, \varepsilon)$ for all $q \in Q$.

Def: A subset $T \subseteq Q$ is called **ε-closed** if there is no ε-transition from any state in T to any state outside T. ◀

Evidently, $\emptyset$ and Q are always ε-closed.

Lemma: ε-closed subsets are closed under intersection.

Proof Let T_1 and T_2 be two ε-closed subsets, and $T = T_1 \cap T_2$. If $T = \emptyset$, the result is obvious. So assume that $T \neq \emptyset$. Suppose that T is not ε-closed. Then, there exists $t \in T$ and $t' \in Q - T$ such that $t \to t'$ is an ε-transition. Since $t' \notin T_1 \cap T_2$, it follows that either $t' \notin T_1$ or $t' \notin T_2$ (or both). But then, if $t' \notin T_1$, then T_1 is not ε-closed, whereas if $t' \notin T_2$, then T_2 is not ε-closed, a contradiction. ◀

1. (a) Prove that ε-closed subsets are closed under union.
(b) Prove that ε-closed subsets are non necessarily closed under complement.

2. Let T be an ε-closed subset of Q. Prove that $\bigcup_{t \in T} \Delta(t, \varepsilon) = T$.

Def: Let $T \subseteq Q$. The **ε-closure** of T is the smallest (with respect to containment) subset C of Q such that C is ε-closed, and $T \subseteq C$. ◀

By the above lemma (and the fact that Q itself is ε-closed), the ε-closure of any subset of Q is uniquely defined. We denote the ε-closure of T as $\varepsilon\text{-closure}(T)$.

3. Let T_1 and T_2 be two subsets of Q. Prove that $\varepsilon\text{-closure}(T_1 \cup T_2) = \varepsilon\text{-closure}(T_1) \cup \varepsilon\text{-closure}(T_2)$.

We now define the function $\hat{\Delta} : 2^Q \times \Sigma^* \to 2^Q$ as follows. We take all $T \subseteq Q$, $x \in \Sigma^*$, and $a \in \Sigma$ in the following recursive definition.

$$\begin{aligned} \hat{\Delta}(T, \varepsilon) &= \varepsilon\text{-closure}(T), \\ \hat{\Delta}(T, xa) &= \varepsilon\text{-closure}\left(\bigcup_{t \in \hat{\Delta}(T,x)} \Delta(t, a) \right). \end{aligned}$$

Notice that Σ^* consists of strings containing symbols in Σ alone. So we take a to be a real symbol (not ε). The ε-transitions are handled by the ε-closures.

Finally, the language of N is defined as

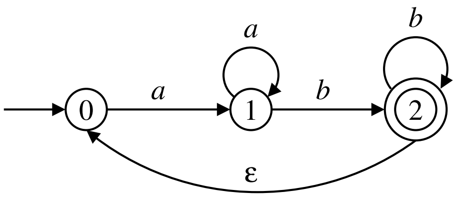
$$\mathscr{L}(N) = \left\{ w \in \Sigma^* \;\middle|\; \hat{\Delta}(S, w) \cap F \neq \emptyset \right\}.$$

We now convert the ε-NFA $N = (Q, \Sigma, \Delta, S, F)$ to an equivalent DFA $D = (Q', \Sigma, \delta, s, F')$. We take Q' to be the set of all ε-closed subsets of Q, $s = \varepsilon\text{-closure}(S)$, and F' to be set of all ε-closed subsets T of Q such that $T \cap F \neq \emptyset$. Finally, for all $T \in Q'$ and $a \in \Sigma$, we define

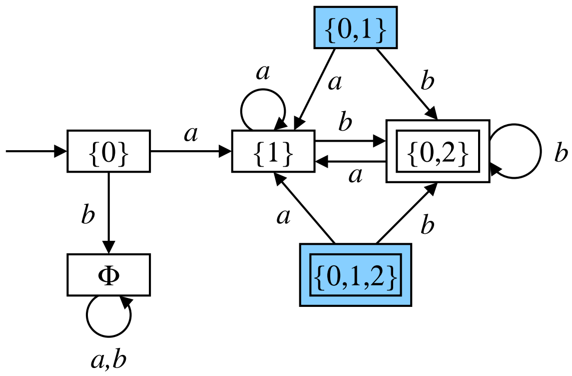
$$\delta(T, a) = \varepsilon\text{-closure}\left(\bigcup_{t \in T} \Delta(t, a) \right).$$

4. Prove that D is a DFA under this transition function δ, and also that $\mathscr{L}(D) = \mathscr{L}(N)$.

Figure 1: Explaining the conversion of an ε-NFA to a DFA



(a) An epsilon–NFA



(b) The converted DFA

As an example, the ε-NFA in Part (a) of Figure 1 has the following transition function. We take $\Sigma = \{a,b\}$.

$$\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
\Delta(0,a) & = & \{1\} & \Delta(0,b) & = & \emptyset & \Delta(0,\varepsilon) & = & \{0\} \\
\Delta(1,a) & = & \{1\} & \Delta(1,b) & = & \{2\} & \Delta(1,\varepsilon) & = & \{1\} \\
\Delta(2,a) & = & \emptyset & \Delta(2,b) & = & \{2\} & \Delta(2,\varepsilon) & = & \{0,2\}
\end{array}$$

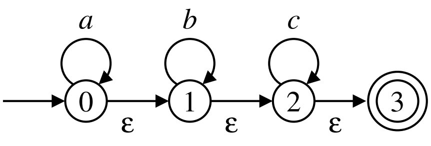
Let $L = \mathscr{L}(N)$. Clearly, L consists all and only the strings of the form $(a^+b^+)^+$, that is,

$$L = \left\{ a^{i_1}b^{j_1}a^{i_2}b^{j_2}\ldots a^{i_n}b^{j_n} \;\middle|\; n \geqslant 1 \text{ and all } i_k \text{ and } j_l \text{ are } \geqslant 1 \right\}.$$

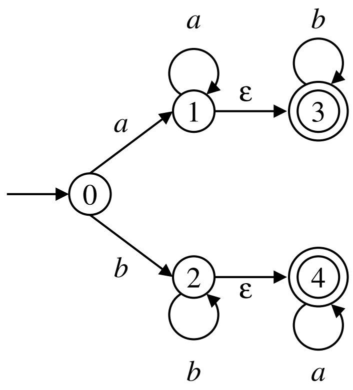
5. Prove that L is the same as the set of all strings over $\{a,b\}$, that start with a and end with b.

$\emptyset$ is always ε-closed. Since there are no outgoing ε-transitions from $0,1$, the subsets $\{0\},\{1\},\{0,1\}$ are ε-closed. Since there is an ε-transition from 2 to 0, any ε-closed subset containing 2 must also contain 0. This gives two more ε-closed subsets $\{0,2\}$ and $\{0,1,2\}$. The subsets $\{2\}$ and $\{1,2\}$ are not ε-closed. So we take $Q' = \{\emptyset,\{0\},\{1\},\{0,1\},\{0,2\},\{0,1,2\}\}$. We have $s = \varepsilon\text{-closure}(S) = \varepsilon\text{-closure}(\{0\}) = \{0\}$. Since $F = \{2\}$, the final states of D are $\{0,2\}$ and $\{0,1,2\}$ only. In order to illustrate the working of δ, we take the example of $T = \{0,2\}$. We have $\delta(\{0,2\},a) = \varepsilon\text{-closure}(\Delta(0,a) \cup \Delta(2,a)) = \varepsilon\text{-closure}(\{1\} \cup \emptyset) = \varepsilon\text{-closure}(\{1\}) = \{1\}$, and $\delta(\{0,2\},b) = \varepsilon\text{-closure}(\Delta(0,b) \cup \Delta(2,b)) = \varepsilon\text{-closure}(\emptyset \cup \{2\}) = \varepsilon\text{-closure}(\{2\}) = \{0,2\}$. The complete transition diagram is given in Part (b) of the Figure 1. The (ε-closed) states $\{0,1\}$ and $\{0,1,2\}$ are not reachable from the start state $\{0\}$, and can be removed from the converted DFA.

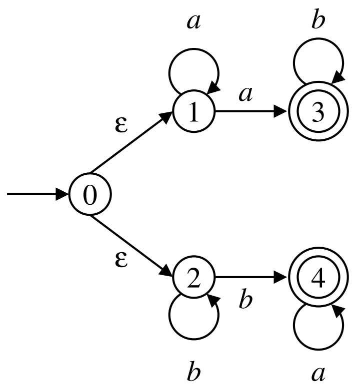
6. Convert the following three ε-NFA to equivalent DFA. In each case, mark the unreachable states (if any).



$\Sigma = \{a,\ b,\ c\}$



$\Sigma = \{\ a,\ b\}$



$\Sigma = \{\ a,\ b\}$

Let us now review the question whether an ε-NFA can be converted to an NFA (without ε-transitions) without invoking the subset-construction procedure. Let us start with the ε-NFA $N = (Q,\Sigma,\Delta,S,F)$. We want to generate an NFA (without ε-transitions) $\tilde{N} = (\tilde{Q}, \Sigma, \tilde{\Delta}, \tilde{S}, \tilde{F})$ with $\mathscr{L}(N) = \mathscr{L}(\tilde{N})$. We take $\tilde{Q} = Q$ and $\tilde{S} = \varepsilon\text{-closure}(S)$. We also take $\tilde{F} = \{q \in Q \mid \varepsilon\text{-closure}(\{q\}) \cap F \neq \emptyset\}$. For every $q \in Q$ and $a \in \Sigma$, we take $\tilde{\Delta}(q,a) = \varepsilon\text{-closure}(\Delta(q,a))$. We do not include any ε-transition of N in $\tilde{\Delta}$. This completes the construction.

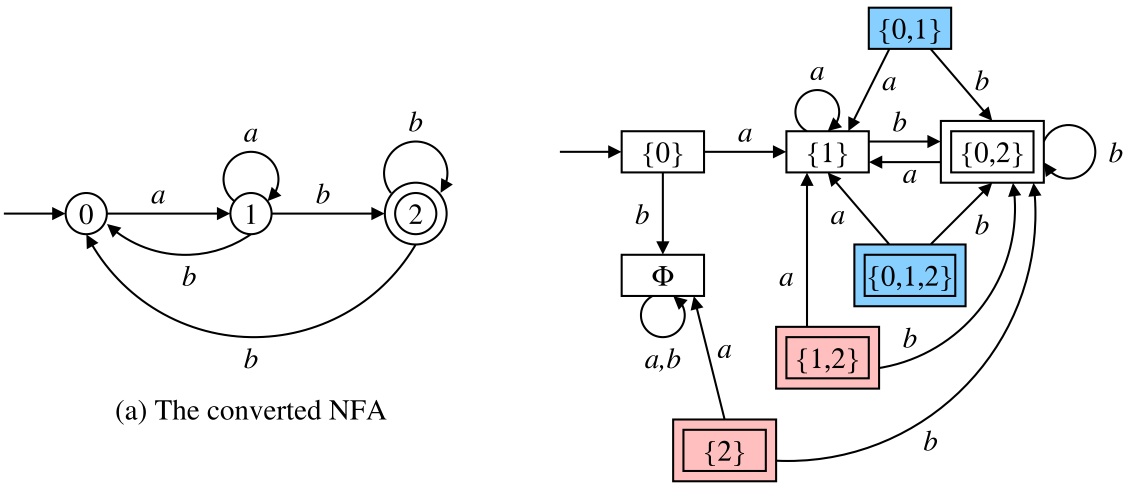
7. We apply the subset-construction procedure on $\tilde{N}$ to generate a DFA $\tilde{D}$.

(a) Prove that all non-ε-closed subsets of Q are unreachable in $\tilde{D}$.

(b) Remove the non-ε-closed subsets from $\tilde{D}$. Prove that after this removal, $\tilde{D}$ becomes exactly the same as the DFA D constructed from the ε-NFA N using the subset-construction procedure described in the text.

(c) Conclude that $\mathscr{L}(\tilde{N}) = \mathscr{L}(N)$.

Figure 2: Explaining the conversion of an ε-NFA to an NFA and then to a DFA



(a) The converted NFA

(b) The converted DFA

Let us illustrate this construction on the ε-NFA of Figure 1. We have $\tilde{S} = \varepsilon\text{-closure}(\{0\}) = \{0\}$. We also have $\varepsilon\text{-closure}(\{1\}) = \{1\}$, and $\varepsilon\text{-closure}(\{2\}) = \{0,2\}$. Therefore $\tilde{F} = \{2\}$. The transition function for $\tilde{N}$ is as follows.

$$
\begin{aligned}
\hat{\Delta}(0,a) &= \varepsilon\text{-closure}(\Delta(0,a)) = \varepsilon\text{-closure}(\{1\}) = \{1\}\\
\hat{\Delta}(0,b) &= \varepsilon\text{-closure}(\Delta(0,b)) = \varepsilon\text{-closure}(\emptyset) = \emptyset\\
\hat{\Delta}(1,a) &= \varepsilon\text{-closure}(\Delta(1,a)) = \varepsilon\text{-closure}(\{1\}) = \{1\}\\
\hat{\Delta}(1,b) &= \varepsilon\text{-closure}(\Delta(1,b)) = \varepsilon\text{-closure}(\{2\}) = \{0,2\}\\
\hat{\Delta}(2,a) &= \varepsilon\text{-closure}(\Delta(2,a)) = \varepsilon\text{-closure}(\emptyset) = \emptyset\\
\hat{\Delta}(2,b) &= \varepsilon\text{-closure}(\Delta(2,b)) = \varepsilon\text{-closure}(\{2\}) = \{0,2\}
\end{aligned}
$$

The converted NFA is given in Part (a) of Figure 2. If we apply the subset-construction procedure on this NFA, we get the DFA in Part (b) of Figure 2. Compare this DFA with the DFA in Part (b) of Figure 1.

8. (a) Apply the ε-NFA-to-NFA construction on the three ε-NFA of Exercise 6.

(b) Apply the subset-construction procedure on each of these constructed NFA, and verify Part (b) of Exercise 7.