SOME DECIDABLE PROBLEMS ABOUT TURING MACHINES

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Properties of RE Sets / Turing Machines?

M accepts ε

- This is a property of RE sets.
- This is specified by any member of $\left\{M \mid \varepsilon \in \mathscr{L}(M)\right\}$.
- This is a non-trivial property, and is undecidable by Rice's theorem, Part 1.
- The property is monotone, so Rice's theorem, Part 2, is not applicable.
- The property is semidecidable.
  - Simulate M on ε, and accept if and only if M accepts.
  - For any M with $\varepsilon \in \mathscr{L}(M)$, the simulation accepts and halts.
  - For an M with $\varepsilon \notin \mathscr{L}(M)$, the simulation may halt in the reject state, or loop.

Properties of RE Sets / Turing Machines?

M halts on ε

- This is not a property of RE sets.
- If $\varepsilon \notin \mathscr{L}(M)$, then upon input ε, M may have a choice to halt after rejecting or loop.
- Rice's theorems cannot be applied in this context.
- The language $\left\{M \mid M \text{ halts on } \varepsilon\right\}$ is r.e. but not recursive.
- RE: Simulate M on ε, and accept if and only if the simulation halts.
- Not recursive: Reduction from HP: $M \# w \mapsto N$.
  - Simulate M on w.
  - If the simulation halts, accept.

Problem 1

Given M, decide whether M contains at least 2021 states.

- A Turing machine looks at the encoding of M, and finds out the answer.
- This machine runs in finite time for every input.

Problem 2

Given M, decide whether M halts within 2021 steps on input ε.

- Simulate M on ε for (at most) 2021 steps.
- If the simulation halts (after accepting/rejecting), accept.
- If the simulation does not halt after 2021 steps, reject.
- This machine is also a decider.

Problem 3

Given M, decide whether M takes more than 2021 steps on some input.

- M takes more than 2021 steps on some input $\iff$
  M takes more than 2021 steps on some input of length $\leqslant 2021$.
- Suppose that M takes $\leqslant 2021$ steps on all inputs of length $\leqslant 2021$. Supply an input w of length > 2021 to M.

- Within 2021 steps, M cannot see more than 2021 symbols from the input.
- This initial behavior of M on w is the same as its behavior on the prefix of w of length 2021. M is deterministic. M halts on w within 2021 steps.
- A decider simulates M on all inputs of length $\leqslant 2021$, each for 2021 steps.
- If some simulation takes more than 2021 steps, accept, else reject.

Problem 4

Given M, decide whether M takes more than 2021 steps on all inputs.

- M takes more than 2021 steps on all inputs $\iff$
  M takes more than 2021 steps on all inputs of length $\leqslant 2021$.
- It suffices to simulate M on all inputs of length $\leqslant 2021$, each for 2021 steps.

Problem 5

Given M, decide whether M ever moves to the right of the 2021-st cell on input ε.

- This problem is semidecidable: Simulate M on ε.
- Let $m = |Q|$ (number of states).
- Let $k = |\Gamma|$ (number of symbols in the tape alphabet).
- Suppose M never goes to the right of the 2021-st cell.
- Total number of configurations possible is $2022mk^{2021}$.
- Simulate M on ε for at most $2022mk^{2021}$ steps.
- If the head ever moves to the right of the 2021-st cell during the simulation, accept.
- If the simulation halts without the head moving to the right of the 2021-st cell, reject.
- Otherwise, some configuration is repeated (pigeon-hole principle).
- Thus the machine must have entered an infinite loop, and will never go beyond the 2021-st cell. Reject.

Three Possibilities

Rice's theorems do not apply to problems specific to Turing machines but not to their languages.

- The problem may be decidable
  - Design a decider for the problem.
  - You must prove that on all inputs, the decider halts after making correct accept/reject decisions.
- The problem may be semidecidable but not decidable.
  - Design a Turing machine (not total) to semidecide the problem.
  - You must prove that all accept decisions are made in finite amounts of time.
  - Use reduction from an undecidable problem like HP.
- The problem may be not even semidecidable.
  - Use reduction from a non-semidecidable problem like $\overline{\text{HP}}$.

Tutorial Exercises

1. Prove that the following problems on a TM M are decidable.
   (a) Decide whether M halts on some input within 2021 steps.
   (b) Decide whether M halts on all inputs within 2021 steps.
   (c) Decide whether M runs for at least 2021^{2021} steps for input a^{2021}.
   (d) Decide whether M on input ε moves left at least ten times.
   (e) Decide whether M on a given input w moves left at least ten times.
2. Is the problem whether a Turing machine on any input reenters the start state decidable or not? Prove.
3. Input: A Turing machine M. Decidable/Semidecidable/Not? Prove.

   (a) M halts on exactly 2021 input strings.
   (b) M halts on at least 2021 input strings.
4. Input: Two Turing machines M and N. Decidable/Semidecidable/Not? Prove.

   (a) M takes more steps than N on input ε.
   (b) M does not take more steps than N on input ε.