CS21201 Discrete Structures

Tutorial 2

Logic

1. While walking in a labyrinth, you find yourself in front of three possible roads. The road on your left is paved with gold, the road in front of you is paved with marble, while the road on your right is made of small stones. Each road is protected by a guard. You talk to the guards, and this is what they tell you.

- The guard of the gold road: "This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."
- The guard of the marble road: "Neither the gold nor the stones will take you to the center."
- The guard of the stone road: "Follow the gold, and you will reach the center. Follow the marble, and you will be lost."

You know that all the guards are liars. Your goal is to choose a correct road that will lead you to the center of the labyrinth. Solve your problem using a propositional-logic formulation and deduction.

Solution Introduce the following propositions.

GG : The guard of the gold road is telling the truth
GM : The guard of the marble road is telling the truth
GS : The guard of the stone road is telling the truth
G : The gold road leads to the center
M : The marble road leads to the center
S : The stone road leads to the center

The statements of the three guards can be logically encoded as follows.

$$\begin{aligned} GG &\leftrightarrow \left[G \wedge (S \rightarrow M)\right] \\ GM &\leftrightarrow \left[\neg G \wedge \neg S\right] \\ GS &\leftrightarrow \left[G \wedge \neg M\right] \end{aligned}$$

You also know that the following statement is true.

$$\begin{aligned} Z &\equiv \neg GG \wedge \neg GM \wedge \neg GS \\ &\equiv \neg\left[G \wedge (S \rightarrow M)\right] \wedge \neg\left[\neg G \wedge \neg S\right] \wedge \neg\left[G \wedge \neg M\right] \\ &\equiv \left[\neg G \vee (S \wedge \neg M)\right] \wedge \left[G \vee S\right] \wedge \left[\neg G \vee M\right]. \end{aligned}$$

The truth table of Z is given below.

G	M	S	$\neg G \vee (S \wedge \neg M)$	$G \vee S$	$\neg G \vee M$	Z
0	0	0	1	0	1	0
0	0	1	1	1	1	1
0	1	0	1	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	1	1	0	0
1	1	0	0	1	1	0
1	1	1	0	1	1	0

This truth table implies that $Z \equiv \left[\neg G \wedge S\right]$, that is, the stone road will surely lead you to the center. (This also implies that the gold road will surely not lead you to the center, and the marble road may or may not lead you to the center, but these conclusions are not part of the solution to your problem.)

2. Prove the following two logical deductions.

$$\begin{array}{c} (\neg p \vee q) \to r \\ r \to (s \vee t) \\ \neg(s \vee u) \\ t \to u \\ q \leftrightarrow v \\ (v \wedge \neg w) \vee (\neg v \wedge w) \to \neg p \\ \hline \therefore\ \neg w \end{array} \qquad \begin{array}{c} t \to q \\ \neg r \to \neg s \\ p \to u \\ \neg t \to \neg r \\ u \to s \\ \hline \therefore\ p \to q \end{array}$$

Solution The first deduction goes as follows.

$$\begin{array}{c} \neg(s \vee u) \\ \hline \therefore\ \neg s\ ,\ \neg u \end{array} \qquad \begin{array}{c} \neg u \\ t \to u \\ \hline \therefore\ \neg t \end{array} \qquad \begin{array}{c} r \to (s \vee t) \\ \neg s\ ,\ \neg t \\ \hline \therefore\ \neg r \end{array} \qquad \begin{array}{c} (\neg p \vee q) \to r \\ \neg r \\ \hline \therefore\ p \wedge \neg q \end{array} \qquad \begin{array}{c} q \leftrightarrow v \\ \neg q \\ \hline \therefore\ \neg v \end{array}$$

$$\begin{array}{c} (v \wedge \neg w) \vee (\neg v \wedge w) \to \neg p \\ p \\ \hline \therefore\ (\neg v \vee w) \wedge (v \vee \neg w) \end{array} \qquad \begin{array}{c} (\neg v \vee w) \wedge (v \vee \neg w) \\ \neg v \\ \hline \therefore\ \neg w \end{array}$$

The second deduction goes as follows.

$$\begin{array}{c} p \to u \\ u \to s \\ \hline \therefore\ p \to s \end{array} \qquad \begin{array}{c} p \to s \\ \neg r \to \neg s \equiv s \to r \\ \hline \therefore\ p \to r \end{array} \qquad \begin{array}{c} p \to r \\ \neg t \to \neg r \equiv r \to t \\ \hline \therefore\ p \to t \end{array} \qquad \begin{array}{c} p \to t \\ t \to q \\ \hline \therefore\ p \to q \end{array}$$

3. Encode and reason about the following.

> *If a scarcity of commodities develops, then the prices rise. If there is a change of government, then fiscal controls will not be continued. If the threat of inflation persists, then fiscal controls will be continued. If there is over-production, then prices do not rise. It has been found that there is over-production and there is a change of government. Therefore, neither the scarcity of commodities has developed, nor there is a threat of inflation.*

Solution The propositions are as follows:

sc : a scarcity of commodities develops
pr : the prices rise
cg : there is a change of government
fc : fiscal controls will be continued
ti : the threat of inflation persists
op : there is over-production

The propositional logic encoding of the statements and logical deduction are as follows:

$$\begin{array}{lcl} F_1 & : & \text{sc} \to \text{pr} \\ F_2 & : & \text{cg} \to \neg\text{fc} \\ F_3 & : & \text{ti} \to \text{fc} \\ F_4 & : & \text{op} \to \neg\text{pr} \\ F_5 & : & \text{op} \wedge \text{cg} \\ \hline G & : & \neg\text{sc} \wedge \neg\text{ti} \end{array}$$

$$\begin{array}{c} \text{cg} \to \neg\text{fc} \\ \text{op} \wedge \text{cg} \\ \hline \therefore\ \neg\text{fc} \end{array} \qquad \begin{array}{c} \text{op} \to \neg\text{pr} \\ \text{op} \wedge \text{cg} \\ \hline \therefore\ \neg\text{pr} \end{array} \qquad \begin{array}{c} \text{sc} \to \text{pr} \\ \neg\text{pr} \\ \hline \therefore\ \neg\text{sc} \end{array} \qquad \begin{array}{c} \text{ti} \to \text{fc} \\ \neg\text{fc} \\ \hline \therefore\ \neg\text{ti} \end{array}$$

4. Your task is to (logically) solve a murder mystery on behalf of Sherlock Holmes, which appeared in the novel "A Study in Scarlet" by Sir Arthur Conan Doyle. The arguments (simplified from the novel) go as follows.

1. There was a murder. If it was not done for robbery, then either it was a political assassination, or it might be for a woman.
2. In case of robbery, usually something is taken.
3. However, nothing was taken from the murderer's place.
4. Political assassins leave the place immediately after their assassination work gets completed.
5. On the contrary, the assassin left his/her tracks all over the murderer's place.
6. For an assassin, to leave tracks all over the murderer's place indicates that (s)he was there all the time (for long duration).

Logically deduce the reason for the murder.

Solution We may use the following propositions.

rob	:	The murder was done for robbery.
pol	:	The murder was a political assassination.
wom	:	The murder was for a woman.
tak	:	Something was taken from the murderer's place.
imm	:	The assassin left immediately after work done.
trc	:	The assassin left tracks all over the room.

The propositional-logic formulas to encode the *six* statements above are given now.

(1) $\neg\text{rob} \to \text{pol} \vee \text{wom}$
(2) $\text{rob} \to \text{tak}$
(3) $\neg\text{tak}$
(4) $\text{pol} \to \text{imm}$
(5) trc
(6) $\text{trc} \to \neg\text{imm}$

The deduction steps are as follows.

$$\begin{array}{l} \text{rob} \to \text{tak} \\ \neg\text{tak} \\ \hline \therefore \neg\text{rob} \end{array} \quad \text{(Modus Tollens)}$$

$$\begin{array}{l} \neg\text{rob} \\ \neg\text{rob} \to \text{pol} \vee \text{wom} \\ \hline \therefore \text{pol} \vee \text{wom} \end{array} \quad \text{(Modus Ponens)}$$

$$\begin{array}{l} \text{trc} \\ \text{trc} \to \neg\text{imm} \\ \hline \therefore \neg\text{imm} \end{array} \quad \text{(Modus Ponens)}$$

$\neg$imm
pol $\rightarrow$ imm

$\therefore \neg$pol (Modus Tollens)

pol $\vee$ wom
$\neg$pol

$\therefore$ wom (Disjunctive Syllogism)

It follows that the murder was done for a woman.

5. Let $p(x)$ be an open statement. Encode the following as quantified expressions.

(a) $p(x)$ is true for exactly one value of x.

Solution $\exists x\Big[p(x) \wedge \forall y\big[p(y) \rightarrow (y = x)\big]\Big]$

(b) $p(x)$ is true for exactly two values of x.

Solution $\exists x \exists y\Big[p(x) \wedge p(y) \wedge (x \neq y) \wedge \forall z\big[p(z) \rightarrow (z = x) \vee (z = y)\big]\Big]$

(c) $p(x)$ is true for at most one value of x.

Solution $\forall x\big[\neg p(x)\big] \vee \exists x\Big[p(x) \wedge \forall y\big[p(y) \rightarrow (y = x)\big]\Big]$

(d) $p(x)$ is true for at least two values of x.

Solution $\exists x \exists y\Big[p(x) \wedge p(y) \wedge (x \neq y)\Big]$

(e) If $p(x)$ is true for at least two values of x, then $p(x)$ is true for all values of x.

Solution $\exists x \exists y\Big[p(x) \wedge p(y) \wedge (x \neq y)\Big] \rightarrow \forall x[p(x)]$

(f) If $p(x)$ is true for at least two values of x, then $p(x)$ is true for at least three values of x.

Solution $\exists x \exists y\Big[p(x) \wedge p(y) \wedge (x \neq y)\Big] \rightarrow \exists x \exists y \exists z\Big[p(x) \wedge p(y) \wedge p(z) \wedge (x \neq y) \wedge (z \neq x) \wedge (z \neq y)\Big]$

6. Prove that $\forall x\big[P(x) \rightarrow (Q(x) \leftrightarrow R(x))\big]$ is equivalent to

$$\Big[\forall x\big[(P(x) \wedge Q(x)) \rightarrow R(x)\big]\Big] \wedge \Big[\forall x\big[(P(x) \wedge R(x)) \rightarrow Q(x)\big]\Big].$$

Solution
$$\begin{aligned}
&\Big[\forall x\big[(P(x) \wedge Q(x)) \rightarrow R(x)\big]\Big] \wedge \Big[\forall x\big[(P(x) \wedge R(x)) \rightarrow Q(x)\big]\Big] \\
&\equiv \forall x\Big[\big[(P(x) \wedge Q(x)) \rightarrow R(x)\big] \wedge \big[(P(x) \wedge R(x)) \rightarrow Q(x)\big]\Big] \\
&\equiv \forall x\Big[\big[\neg P(x) \vee \neg Q(x) \vee R(x)\big] \wedge \big[\neg P(x) \vee \neg R(x) \vee Q(x)\big]\Big] \\
&\equiv \forall x\Big[\neg P(x) \vee \big[(\neg Q(x) \vee R(x)) \wedge (\neg R(x) \vee Q(x))\big]\Big] \\
&\equiv \forall x\Big[P(x) \rightarrow \big[(Q(x) \rightarrow R(x)) \wedge (R(x) \rightarrow Q(x))\big]\Big] \equiv \forall x\big[P(x) \rightarrow (Q(x) \leftrightarrow R(x))\big]
\end{aligned}$$

7. Formalize the following sentences in first-order logic using only the two predicates given below.

(i) inside(x, y) : x is inside of y. (ii) free(x) : x is free.

(a) Something is inside of everything.

Solution $\exists x \forall y(\text{inside}(x, y))$

(b) Everything that is free has nothing inside it.

Solution $\forall x\big[\text{free}(x) \rightarrow \neg \exists y(\text{inside}(y, x))\big]$

(c) There is something that is inside and is not free.

Solution $\exists x\big[\forall y(\text{inside}(x, y)) \wedge \neg \text{free}(x)\big]$

8. Consider the following predicates with the specified meaning.

- $\text{owns}(x,y)$: x owns y
- $\text{bookLover}(x)$: x is a book lover
- $\text{mutilates}(x,y)$: x mutilates y
- $\text{book}(x)$: x is a book
- $\text{kindle}(x)$: x is a kindle

Encode the following statements in the first-order logic using the above predicates.

(a) Tom owns a kindle.

Solution $\exists x \left[\text{owns}(\text{Tom},x) \wedge \text{kindle}(x)\right]$

(b) Every kindle owner loves books.

Solution $\forall x\,\forall y \left[\left(\text{owns}(x,y) \wedge \text{kindle}(y)\right) \to \text{bookLover}(x)\right]$

(c) No book lover mutilates books.

Solution $\forall x\,\forall y \left[\left(\text{bookLover}(x) \wedge \text{book}(y)\right) \to \neg\,\text{mutilates}(x,y)\right]$

(d) Either Tom or Austin mutilates the book called Origin.

Solution $\text{book}(\text{Origin}) \wedge \left(\text{mutilates}(\text{Tom},\text{Origin}) \vee \text{mutilates}(\text{Austin},\text{Origin})\right)$

(e) Every kindle is a book.

Solution $\forall x \left[\text{kindle}(x) \to \text{book}(x)\right]$

9. Encode the following logical statements using predicate logic (formulate suitable predicate and function symbols as required), and conclude on the validity of the last statement.

Every athlete is strong. Everyone who is strong and intelligent will succeed in career. Hima is an athlete. Hima is intelligent. Therefore, Hima will succeed in career.

Solution The predicate logic encoding of the statements are as follows:

$$\begin{aligned}
F_1 &: \forall x\left[\text{athlete}(x) \to \text{strong}(x)\right]\\
F_2 &: \forall x\left[(\text{strong}(x) \wedge \text{intelligent}(x)) \to \text{succeed}(x)\right]\\
F_3 &: \text{athlete}(\text{Hima})\\
F_4 &: \text{intelligent}(\text{Hima})\\
G &: \text{succeed}(\text{Hima})
\end{aligned}$$

The logical deduction for $(F_1 \wedge F_2 \wedge F_3 \wedge F_4) \to G$ is as follows.

$$\frac{\text{athlete}(x) \to \text{strong}(x) \qquad \text{athlete}(\text{Hima})}{\therefore\ \text{strong}(\text{Hima})} \qquad\qquad \frac{(\text{strong}(x) \wedge \text{intelligent}(x)) \to \text{succeed}(x) \qquad \text{strong}(\text{Hima})\ ,\ \ \text{intelligent}(\text{Hima})}{\therefore\ \text{succeed}(\text{Hima})}$$

10. Translate the following into idiomatic English.

(a) $\forall x\Big[\left[H(x) \wedge \forall y \neg M(x,y)\right] \to U(x)\Big]$, where $H(x)$ means x is a man, $M(x,y)$ means x is married to y, $U(x)$ means x is unhappy, and x and y range over people.

Solution All unmarried men are unhappy.

(b) $\exists z \exists x\left[P(z,x) \wedge \forall y(S(z,y) \wedge W(y))\right]$, where $P(z,x)$ means z is a parent of x, $S(z,y)$ means z and y are siblings, $W(y)$ means y is a woman, and x, y, and z range over people.

Solution Some parents have all their siblings as women.

Additional Exercises

11. Encode the following statements using propositions.

> *The system is in a multiuser state iff it is operating normally. If the system is operating normally, the kernel is functioning. Either the kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode.*

Logically deduce that the system is in interrupt mode.

12. Solve the following parts about propositions. Take $n \geqslant 2$.

(a) Prove that $p_1 \vee p_2 \vee \cdots \vee p_n \to q$ is equivalent to $(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q)$.

(b) Prove that $p_1 \wedge p_2 \wedge \cdots \wedge p_n \to q$ is not equivalent to $(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q)$. If there is only a one-way implication, prove it.

(c) Prove that $p \to q_1 \vee q_2 \vee \cdots \vee q_n$ is equivalent to $(p \to q_1) \vee (p \to q_2) \vee \cdots \vee (p \to q_n)$.

(d) Prove that $p \to q_1 \wedge q_2 \wedge \cdots \wedge q_n$ is equivalent to $(p \to q_1) \wedge (p \to q_2) \wedge \cdots \wedge (p \to q_n)$.

(e) Prove that $p \to q_1 \vee q_2 \vee \cdots \vee q_n$ is equivalent to $(p \wedge \neg q_1 \wedge \neg q_2 \wedge \cdots \wedge \neg q_{n-1}) \to q_n$.

(f) Is $(p \to q) \to r$ equivalent to $p \to (q \to r)$?

(g) Is $(p \leftrightarrow q) \leftrightarrow r$ equivalent to $p \leftrightarrow (q \leftrightarrow r)$?

13. Prove that the negation of the statement $\exists x \forall y \big[P(x,y) \to \neg Q(y)\big]$ is $\forall x \exists y \big[P(x,y) \wedge Q(y)\big]$.

14. Formalize the following sentences in first-order logic using only the two predicates given below.

(i) $\text{love}(x,y)$: x loves y. (ii) $\text{diff}(x,y)$: x differs from y.

(a) There is exactly one person who loves Mary.

(b) Only Bob loves Mary.

(c) There are exactly two persons who love Mary.

(d) There are at most two persons who love Mary.

(e) If Bob loves everyone that Mary loves, and Bob loves David, then Mary does not love David.

15. Encode the following logical statements using predicate logic (formulate suitable predicate and function symbols as required), and conclude on the validity of the last statement.

(a) *No man who is a candidate will be defeated if he is a good campaigner. Any man who runs for office is a candidate. Any candidate who is not defeated will be elected. Every man who is elected is a good campaigner. Therefore, Any man who runs for office will be elected if and only if he is a good campaigner.*

(b) *Jack owns a dog. Every dog owner is an animal lover. No animal lover kills an animal. Either Jack or Curiosity killed Tuna, which is a cat. Did Curiosity kill the cat?*

(c) *All members are both officers and gentlemen. All officers are fighters. Only a pacifist is a gentleman or not a fighter. No pacifist is a gentleman if he is a fighter. Some members are fighters iff they are officers. Therefore, not all members are fighters.*

(d) *Everything that dies decomposes. Everything that decomposes divides into parts. Only material things are divisible into parts. No human souls are material. Therefore, no human souls die.*

16. For each part, if the formula is valid, show a proof in natural deduction. If not, provide a counter-example.

(a) $\forall x \big[P(x) \to \exists y P(y)\big]$

(b) $\exists x \big[P(x) \to \forall y P(y)\big]$

(c) $\neg\neg\forall x \big[P(x)\big] \to \forall x \big[\neg\neg P(x)\big]$

(d) $\Big[\forall x \big[P(x) \to \exists y Q(x,y)\big]\Big] \to \Big[\exists x P(x) \to \exists y Q(x,y)\Big]$

(e) $\Big[\exists x Q(x) \wedge \big[\forall x (P(x) \to \neg Q(x))\big]\Big] \to \exists x \big[\neg P(x)\big]$