	INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR	
		Stamp / Signature of the Invigilator

EXAMINATION (Mid Semester)	**SEMESTER (Autumn)**

Roll Number										**Section**		**Name**	
Subject Number	C	S	2	1	2	0	1	**Subject Name**				***Discrete Structures***	
Department / Center of the Student											**Additional sheets**		

Important Instructions and Guidelines for Students

1. You must occupy your seat as per the Examination Schedule/Sitting Plan.
2. Do not keep mobile phones or any similar electronic gadgets with you even in the switched off mode.
3. Loose papers, class notes, books or any such materials must not be in your possession, even if they are irrelevant to the subject you are taking examination.
4. Data book, codes, graph papers, relevant standard tables/charts or any other materials are allowed only when instructed by the paper-setter.
5. Use of instrument box, pencil box and non-programmable calculator is allowed during the examination. However, exchange of these items or any other papers (including question papers) is not permitted.
6. Write on both sides of the answer script and do not tear off any page. **Use last page(s) of the answer script for rough work.** Report to the invigilator if the answer script has torn or distorted page(s).
7. It is your responsibility to ensure that you have signed the Attendance Sheet. Keep your Admit Card/Identity Card on the desk for checking by the invigilator.
8. You may leave the examination hall for wash room or for drinking water for a very short period. Record your absence from the Examination Hall in the register provided. Smoking and the consumption of any kind of beverages are strictly prohibited inside the Examination Hall.
9. Do not leave the Examination Hall without submitting your answer script to the invigilator. **In any case, you are not allowed to take away the answer script with you.** After the completion of the examination, do not leave the seat until the invigilators collect all the answer scripts.
10. During the examination, either inside or outside the Examination Hall, gathering information from any kind of sources or exchanging information with others or any such attempt will be treated as **'unfair means'**. Do not adopt unfair means and do not indulge in unseemly behavior.

Violation of any of the above instructions may lead to severe punishment.

Signature of the Student

To be filled in by the examiner

Question Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Marks Obtained											

Marks obtained (in words)	**Signature of the Examiner**	**Signature of the Scrutineer**

CS21201/CS21001 Discrete Structures, Autumn 2022–2023

23-Sep-2022, 09:00am–11:00am **Mid-Semester Test** Maximum marks: 60

Instructions

- Write your answers in the question paper itself. Be brief and precise. Answer all questions.
- Write the answers only in the respective spaces provided. The last three blank pages may be used for rough work.
- If you use any theorem/result/formula covered in the class, just mention it, do not elaborate.
- Write all the proofs in mathematically precise language. Unclear and/or dubious statements would be severely penalized.

Do not write anything on this page.

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1. Let $P(x)$ and $Q(y)$ be open statements with the variables x and y coming from their respective non-empty universes. Using only a counterexample, establish the claim in each of the following two parts. There is no need to mention how you came up with the counterexample. Mention only the universes for x and y, what the open statements $P(x)$ and $Q(y)$ are, and how this example establishes logical non-equivalence.

(a) $\forall x[P(x)] \to \forall y[Q(y)]$ is logically not equivalent to $\forall x\Big[P(x) \to \forall y[Q(y)]\Big]$. **(5)**

Solution Let the universes of x and y be the set of positive integers, $P(x)$ the open statement "x is even", and $Q(y)$ the open statement "y is odd". But then, both $\forall x[P(x)]$ and $\forall y[Q(y)]$ are false, so the implication $\forall x[P(x)] \to \forall y[Q(y)]$ is true. But for any even x, the implication $P(x) \to \forall y[Q(y)]$ is false, and so $\forall x\Big[P(x) \to \forall y[Q(y)]\Big]$ is false.

(b) $\exists x[P(x)] \to \exists y[Q(y)]$ is logically not equivalent to $\exists x\Big[P(x) \to \exists y[Q(y)]\Big]$. **(5)**

Solution Let again the universe for x and y be the set of positive integers. Take $P(x)$ to be the open statement "x is even", and $Q(y)$ to be the open statement "both y and $y+1$ are odd". But then, $\exists x[P(x)]$ is true, and $\exists y[Q(y)]$ is false, so the implication $\exists x[P(x)] \to \exists y[Q(y)]$ is false. On the other hand, for any odd x, the implication $P(x) \to \exists y[Q(y)]$ is true, that is, $\exists x\Big[P(x) \to \exists y[Q(y)]\Big]$ is true.

2. Let n be a non-negative integer. Consider all paths in the grid from $(0,0)$ to (n,n), that never go above the main diagonal $y = x$, and that consist of the following three types of moves:

$R:$ Move one step right (from (x,y) to $(x+1,y)$),

$U:$ Move one step up (from (x,y) to $(x,y+1)$), and

$D:$ If the current position is on the main diagonal, then move along the diagonal for one step (from (x,x) to $(x+1,x+1)$). A diagonal move from (x,y) to $(x+1,y+1)$ is not allowed if $x \neq y$.

Let $D(n)$ be the number of such paths from $(0,0)$ to (n,n). Prove that $D(n) = C(n+1)$, where $C(n+1)$ is the $(n+1)$-th Catalan number. (**Hint:** First derive a recurrence for $D(n)$. Then proceed by induction on n.) **(10)**

Solution Consider a path P of the given type. If P never takes a diagonal move, there are $C(n)$ possibilities. Otherwise P makes the first diagonal move at (k,k) for some $k \in \{0,1,2,\ldots,n-1\}$. There are $C(k)$ possibilities for reaching from $(0,0)$ to (k,k) (no diagonal movements up to this time). Then the diagonal move takes the path to $(k+1,k+1)$. From there, we can reach (n,n) in $D(n-k-1)$ ways. We therefore have

$$D(n) = C(n) + \sum_{k=0}^{n-1} C(k)D(n-1-k).$$

We now proceed by strong induction on n. The basis case corresponds to $n = 0$. It is easy to see that $D(0) = 1$ and $C(1) = 1$. So take some $n \geqslant 1$, and assume that $D(i) = C(i+1)$ for all $i = 1,2,3,\ldots,n-1$. But then, the above recursion gives

$$D(n) = C(n) + \sum_{k=0}^{n-1} C(k)C(n-1) = C(n)C(0) + \sum_{k=0}^{n-1} C(k)C(n-k) = \sum_{k=0}^{n} C(k)C(n-k) = C(n+1).$$

In these calculations, we use the fact that $C(0) = 1$. We also use the recurrence for Catalan numbers.

3. In the following C function, you supply two non-negative integers a, b as arguments.

```
int f ( int a, int b )
{
    int x, y, t;

    x = a; y = b; t = 0;
    while (x > 0) {
        --x; t -= x;
        t += y; ++y;
    }
    return t;
}
```

Using an invariance (involving the variables a, b, x, y, t only) of the loop in the function, determine what $f(a, b)$ returns as a function of a and b. Give a closed-form formula. There is no need to mention how you came up with the invariance. Instead show clearly (a) what the invariance is, (b) initially the invariance is true, (c) the loop maintains the invariance, and (d) how the invariance gives the formula for the return value. No credit for any deduction not based on loop invariance. **(10)**

Solution The loop maintains the invariance

$$t + xy = ab.$$

This is true initially: $0 + ab = ab$. Suppose that at the start of one iteration, we have $t + xy = ab$. In the loop body, x changes to $x - 1$, t changes to $t - (x - 1) + y$, and y changes to $y + 1$. We therefore have

$$[t - (x - 1) + y] + (x - 1)(y + 1) = t - x + y + 1 + xy + x - y - 1 = t + xy = ab.$$

The loop terminates when $x = 0$. At that time, the invariance $t + xy = ab$ implies $t = ab$.

4. Let $n \geqslant 2$ be an integer. You choose n distinct integers from the set $\{1, 2, 3, \ldots, n^2 - 1\}$. Prove that there must be two of the chosen integers (call them x and y) satisfying $0 < \sqrt{x} - \sqrt{y} < 1$. **(10)**

Solution This follows from a direct application of the pigeon-hole principle. The $n-1$ holes are

$$\begin{aligned}
&\{1,2,3\},\\
&\{4,5,6,7,8\},\\
&\{9,10,11,12,13,14,15\},\\
&\ldots,\\
&\{(n-1)^2,(n-1)^2+1,(n-1)^2+2,\ldots,n^2-1\}.
\end{aligned}$$

The pigeons are the n chosen integers.

5. Let A be a set, and f, g, h three functions $A \to A$ such that $h \circ g \circ f$ is the identity function 1_A of A. Assume that neither of f, g, h is the identity function of A.

(a) Prove that f must be injective (one-one), and h must be surjective (onto). **(4)**

Solution Take $a_1, a_2 \in A$. If $f(a_1) = f(a_2)$, then $(h \circ g)(f(a_1)) = (h \circ g)(f(a_2))$, that is, $a_1 = a_2$.

Then, take any $b \in A$. We have $(h \circ g \circ f)(b) = b$, But then, $h(a) = b$, where $a = g(f(b))$.

(b) Demonstrate by an explicit example that g may be neither injective nor surjective. Do not just argue that such a g is possible. (**Hint:** A must be an infinite set.) **(6)**

Solution Take $A = \mathbb{N}$ (the set of positive integers). Take the three functions as follows.

- $f(a) = a+2$ for all $a \in \mathbb{N}$.
- $g(1) = g(2) = 2$, and $g(a) = a-1$ for all $a \geqslant 3$.
- $h(1) = 1$, and $h(a) = a-1$ for all $a \geqslant 2$.

Neither of f, g, h is the identity function. For all $a \in \mathbb{N}$, we have $f(a) = a+2 \geqslant 3$, so $g(f(a)) = (a+2)-1 = a+1 \geqslant 2$, and therefore $h(g(f(a))) = (a+1)-1 = a$. The function g is not injective, because $g(1) = g(2)$. It is not surjective too, because $1 \notin g(\mathbb{N})$.

6. Let A be a (non-empty) set, and E the set of all equivalence relations on A. Take two equivalence relations R_1 and R_2 from E. We say that R_1 refines R_2 if $x\,R_1\,y$ implies $x\,R_2\,y$ for all $x, y \in A$.

(a) Prove that E is a poset under refinement. (**Hint:** Every binary relation on A is a subset of $A \times A$.) **(6)**

Solution Every binary relation on A is a subset of $A \times A$. Therefore R_1 refines R_2 if and only if $R_1 \subseteq R_2$.

Let $R, R_1, R_2, R_3 \in E$.

[Reflexive] $R \subseteq R$.

[Antisymmetric] Let R_1 refine R_2, and R_2 refine R_1. But then, $R_1 \subseteq R_2$ and $R_2 \subseteq R_1$. Therefore $R_1 = R_2$.

[Transitive] Let R_1 refine R_2, and R_2 refine R_3. This means $R_1 \subseteq R_2$ and $R_2 \subseteq R_3$. But then, $R_1 \subseteq R_3$, that is, R_1 refines R_3.

(b) Does E always contain a least element under refinement? If so, what is it, and why? If not, supply an explicit counterexample. **(4)**

Solution The equality relation is an equivalence relation. Moreover, every equivalence relation R is reflexive. Therefore $x = y$ implies $x\,R\,y$, that is, $=$ refines R. Consequently, $=$ is the least element of E.