Divide-and-Conquer Recurrences

- General form: $T_N = aT_{N/b} + cN^d$.
- Involves floor and ceiling because N/b need not be an integer for all N.
- Examples
  - Binary search: $a = 1$, $b = 2$, $d = 0$.
  - Merge sort: $a = 2$, $b = 2$, $d = 1$.
  - Strassen's matrix multiplication: $a = 7$, $b = 2$, $d = 2$.
- This does not look like a recurrence of finite order.
- We can convert it to a recurrence of finite order for specific values of N.
- We assume that T_N is increasing, that is, $M \leqslant N$ implies $T_M \leqslant T_N$.
- We can supply a big-Θ bound for T_N for a general N.

The Master Theorem

- $T_N = aT_{N/b} + cN^d$ whenever N is a power of b.
- Assume that T_N is an increasing function of N. This is usually true, and can be proved by (strong) induction on N using the recurrence (with floors and/or ceilings).
- Let $\delta = \log_b a$.
- Then, we have

$$T_N = \begin{cases} \Theta(N^\delta) & \text{if } \delta > d, \\ \Theta(N^d) & \text{if } d > \delta, \\ \Theta(N^d \log N) = \Theta(N^\delta \log N) & \text{if } d = \delta. \end{cases}$$

Proof for the Special Case

- Let $N = b^n$.
- Define $t_n = T_{b^n}$.
- The recurrence gives

$$t_n = a t_{n-1} + c s^n,$$

where

$$s = b^d.$$

- The characteristic equation is $r - a = 0$, that is, $r = a$.
- The solution of this recurrence is of the form

$$t_n = \begin{cases} A a^n + U s^n & \text{if } s \neq a, \\ A a^n + U n s^n & \text{if } s = a, \end{cases}$$

where A and U are constants.

Proof for the Special Case

- Case 1: $a > s$
  - $a > b^d$ implies $\log_b a > d$, that is, $\delta > d$.
  - $T_N = t_n = \Theta(a^n)$.
  - $a^n = b^{n \log_b a} = b^{n\delta} = N^\delta$.
- Case 2: $a < s$
  - $a < b^d$ implies $\log_b a < d$, that is, $\delta < d$.
  - $T_N = t_n = \Theta(s^n)$.
  - $s^n = (b^d)^n = (b^n)^d = N^d$.
- Case 3: $a = s$
  - $a = b^d$ implies $\log_b a = d$, that is, $\delta = d$.
  - $a^n = s^n$.
  - $T_N = t_n = \Theta(ns^n)$.
  - $ns^n = nN^d = N^d \log_b N = N^\delta \log_b n$.

Proof for the General Case

- Take $b^{n-1} < N \leqslant b^n$.
- By the increasing property, we have $T_{b^{n-1}} \leqslant T_N \leqslant T_{b^n}$.
- As an example, consider Case 1.
- $T_N \leqslant T_{b^n}$, and $(b^n)^\delta < (Nb)^\delta = N^\delta b^\delta$.
- $T_N \geqslant T_{b^{n-1}}$, and $(b^{n-1})^\delta = (b^n)^\delta / b^\delta \geqslant N^\delta / b^\delta$.
- b^δ is a constant, so $T_N = \Theta(N^\delta)$.