Set Theory

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Autumn 2020

Sets and Subsets: Definitions and Properties

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Note: $|\phi| = 0$, but $\phi \neq \{0\}$ and $\phi \neq \{\phi\}$ (since, $|\{0\}| = |\{\phi\}| = 1$)

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Frequently-Used Set Examples and Notations

Popular Set Examples:

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\mathbb{Z} &= \text{Set of Integers} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbb{Z}^+ &= \text{Set of Positive Integers} = \{x \in \mathbb{Z} \mid x > 0\} \\
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\mathbb{C}^* &= \text{Set of Non-zero Complex Numbers} = \{c \in \mathbb{C} \mid c \neq 0\}
\end{aligned}
$$

Frequently-Used Notations:

- For each $n \in \mathbb{Z}^+$, $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$
- For real numbers, a, b with $a < b$, we define intervals as follows:

  (Closed) $[a, b] = \{x \mid a \leq x \leq b\}$ (Open) $(a, b) = \{x \mid a < x < b\}$

  (Half-Open) $(a, b] = \{x \mid a < x \leq b\}$ and $[a, b) = \{x \mid a \leq x < b\}$

Counting using Set Theory

Prove that, $\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$

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Ex: Let $n = 4$ and $\mathcal{S} = \{1, 2, 3, 4\}$. All 2-element subsets are, $\mathcal{A}_1 = \{1, 2\}$, $\mathcal{A}_2 = \{1, 3\}$, $\mathcal{A}_3 = \{1, 4\}$, $\mathcal{A}_4 = \{2, 3\}$, $\mathcal{A}_5 = \{2, 4\}$, $\mathcal{A}_6 = \{3, 4\}$. From each $\mathcal{A}_i$s, a 3-element subset can be formed in *two* ways. So, total possibilities $= 2 \times \binom{4}{2} = 12$.

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Ex: 3-element subset $\{1, 2, 3\}$ can formed from $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4$ by adding an element to each. So, reduced number of possibilities $= \frac{12}{3} = 4 = \binom{4}{3}$

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Let the $(n+1)$-element set be $= \{1,2,\ldots,n,n+1\}$ From $(n+1)$-element set, choosing $(r+1)$-element subsets with smallest element i can be done in $\binom{n+1-i}{r}$ ways. So, all such possible choice leads to, $\sum_{i=1}^{(n+1)-(r+1)} \binom{n+1-i}{r} = \binom{n+1}{r+1}$, implying the proof.

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Prove that, $\sum_{i=0}^{n} i\binom{n}{i} = n.2^{n-1}$

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Consider, $n = 4$.

Summand	Subset Correspondence
$1+1+1+1 = 1+1+1+1$	ϕ
$2+1+1 = (1+1)+1+1$	$\{1\}$
$1+2+1 = 1+(1+1)+1$	$\{2\}$
$1+1+2 = 1+1+(1+1)$	$\{3\}$
$3+1 = (1+1+1)+1$	$\{1,2\}$
$2+2 = (1+1)+(1+1)$	$\{1,3\}$
$1+3 = 1+(1+1+1)$	$\{2,3\}$
$4 = (1+1+1+1)$	$\{1,2,3\}$

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$4 = (1+1+1+1)$	$\{1,2,3\}$

$\therefore$ Number of summands of n = Number of subsets of an $(n-1)$-element set $= 2^{n-1}$.

Set Operations

For two sets, $\mathcal{A}, \mathcal{B} \in \mathcal{U}$ (universal set), the following operations are defined:

(Ex: Let, $\mathcal{A} = \{1, 2, 3\}$ and $\mathcal{B} = \{2, 3, 4\}$)

Union: $\mathcal{A} \cup \mathcal{B} = \{x \mid x \in \mathcal{A} \vee x \in \mathcal{B}\}$ (Ex: $\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4\}$)

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Symmetric Difference:

$$\begin{aligned}
\mathcal{A} \,\Delta\, \mathcal{B} &= \{x \mid (x \in \mathcal{A} \vee x \in \mathcal{B}) \wedge x \notin \mathcal{A} \cap \mathcal{B}\} \\
&= \{x \mid x \in \mathcal{A} \cup \mathcal{B} \wedge x \notin \mathcal{A} \cap \mathcal{B}\} = (\mathcal{A} \cup \mathcal{B}) - (\mathcal{A} \cap \mathcal{B}) \\
&= \{x \mid x \in \mathcal{A} \cap \overline{\mathcal{B}} \wedge x \in \overline{\mathcal{A}} \cap \mathcal{B}\} = (\mathcal{A} \cap \overline{\mathcal{B}}) \cup (\overline{\mathcal{A}} \cap \mathcal{B}) \\
&= (\mathcal{A} - \mathcal{B}) \cup (\mathcal{B} - \mathcal{A}) = \mathcal{B} \,\Delta\, \mathcal{A}
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Mutual Disjoint: Sets, $\mathcal{A}$ and $\mathcal{B}$, are mutually disjoint (or disjoint), when $\mathcal{A} \cap \mathcal{B} = \phi$.
In such a case, $\mathcal{A} \,\Delta\, \mathcal{B} = \mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \overline{\mathcal{B}} = \mathcal{A}$ and $\overline{\mathcal{A}} \cap \mathcal{B} = \mathcal{B}$.

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The following statements are equivalent: *(Proof Left as an Exercise!)*

(a) $\mathcal{A} \subseteq \mathcal{B}$, (b) $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$, (c) $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$, (d) $\overline{\mathcal{B}} \subseteq \overline{\mathcal{A}}$

Laws of Set Theory

For three sets, $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}$, the rules given as follows:

Name of the Law	Mathematical Expressions
Double Complement:	$\overline{\overline{\mathcal{A}}} = \mathcal{A}$
DeMorgan's Laws:	$\overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$, $\overline{\mathcal{A} \cap \mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$
Commutative Laws:	$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$, $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$
Associative Laws:	$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$, $\mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}$
Distributive Laws:	$\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$, $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$
Idempotent Laws:	$\mathcal{A} \cup \mathcal{A} = \mathcal{A}$, $\mathcal{A} \cap \mathcal{A} = \mathcal{A}$
Identity Laws:	$\mathcal{A} \cup \phi = \mathcal{A}$, $\mathcal{A} \cap \mathcal{U} = \mathcal{A}$
Inverse Laws:	$\mathcal{A} \cup \overline{\mathcal{A}} = \mathcal{U}$, $\mathcal{A} \cap \overline{\mathcal{A}} = \phi$
Domination Laws:	$\mathcal{A} \cup \mathcal{U} = \mathcal{U}$, $\mathcal{A} \cap \phi = \phi$
Absorption Laws:	$\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}$, $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$

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An Example Proof Sketch: $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$

$$x \in \mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) \Leftrightarrow (x \in \mathcal{A}) \vee (x \in \mathcal{B} \cap \mathcal{C}) \Leftrightarrow (x \in \mathcal{A}) \vee ((x \in \mathcal{B}) \wedge (x \in \mathcal{C}))$$
$$\Leftrightarrow ((x \in \mathcal{A}) \vee (x \in \mathcal{B})) \wedge ((x \in \mathcal{A}) \vee (x \in \mathcal{C}))$$
$$\Leftrightarrow (x \in \mathcal{A} \cup \mathcal{B}) \wedge (x \in \mathcal{A} \cup \mathcal{C}) \Leftrightarrow x \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$$

Some Derived Laws and Observations

$$\mathcal{A}_1 = \mathcal{A}_1 \cup (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4) \cup \cdots \qquad (\forall i,\ \mathcal{A}_i \in \mathcal{U})$$

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Proof: $\mathcal{A}_1 \cup (\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{A}_1, \quad (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) = (\mathcal{A}_1 \cap \mathcal{A}_2),$
$(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4) = (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3)$, *and so on ...*

Some Derived Laws and Observations

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Proof: $\mathcal{A}_1 \cup (\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{A}_1, \quad (\mathcal{A}_1 \cap \mathcal{A}_2) \cup (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) = (\mathcal{A}_1 \cap \mathcal{A}_2),$
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Index Set and Partitions

Index Set

Definition: Let $\mathcal{I} \neq \phi$ and $\forall i \in \mathcal{I}$, let $\mathcal{A}_i \subseteq \mathcal{U}$ (universal set). Then, $\mathcal{I}$ is called an *index set*, and each $i \in \mathcal{I}$ is an index.

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Set Operations: (Union) $\bigcup_{i \in \mathcal{I}} \mathcal{A}_i = \{x \mid \exists i \in \mathcal{I},\ x \in \mathcal{A}_i\}$
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Partition of a Set

Definition: Let $\mathcal{S}$ be a non-empty set. A family of non-empty subsets, $\{\mathcal{S}_i \mid i \in \mathcal{I}\}$ ($\mathcal{I}$ being the index set) is said to form a partition of $\mathcal{S}$ if the following two condition holds:

- $\bigcup_{i \in \mathcal{I}} \mathcal{S}_i = \mathcal{S}$ (Complete Set Cover), and
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Example: Let $\mathcal{Z}_0 = \{3m \mid m \text{ is an integer}\} = \{0, \pm 3, \pm 6, \ldots\}$,
$\mathcal{Z}_1 = \{3m + 1 \mid m \text{ is an integer}\} = \{\ldots, -8, -5, -2, +1, +4, +7, \ldots\}$
$\mathcal{Z}_2 = \{3m + 2 \mid m \text{ is an integer}\} = \{\ldots, -7, -4, -1, +2, +5, +8, \ldots\}$
Now, $\mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \mathcal{Z}_2 = \mathbb{Z}$ and $\mathcal{Z}_0 \cap \mathcal{Z}_1 = \mathcal{Z}_1 \cap \mathcal{Z}_2 = \mathcal{Z}_2 \cap \mathcal{Z}_0 = \phi$

Thank You!