Relations

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Cartesian Product

Definition: Cartesian Product or Cross Product of two sets, $\mathcal{A}$ and $\mathcal{B}$, denoted as $\mathcal{A} \times \mathcal{B}$, is defined by, $\mathcal{A} \times \mathcal{B} = \{(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}$

Generically, $\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k = \{(x_1, x_2, \ldots, x_k) \mid \forall i,\ x_i \in \mathcal{A}_i\}$

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Relations and Examples

(Binary) Relation

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Example

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$$\mathcal{B} \times \mathcal{A} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

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There can be a total of $2^6 = 64$ different (binary) relations possible. Some are:

$\rho_1 = \{(1, a), (1, b), (1, c)\}$ or $\rho_2 = \{(2, a), (3, a), (1, b), (3, b)\}$.

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Let a relation, ρ, is defined over the set, $\mathcal{A}$ with $|\mathcal{A}| = n$, as $\rho \subseteq \mathcal{A} \times \mathcal{A}$. (Count: 2^{n^2})

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5. *Only Reflexive*: Relation $\rho = \{(A, B) \mid \text{Person-}A \text{ knows Person-}B\}$

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Examples of Relations

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Equivalence Relation and Equivalence Classes

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- Reflexive since $(x - x) = 0$ is divisible by 5.
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- $[0] = \{\ldots, -6, -3, 0, +3, +6, \ldots\} = \{3k \mid k \in \mathbb{Z}\}$
- $[1] = \{\ldots, -5, -2, 1, +4, +7, \ldots\} = \{3k + 1 \mid k \in \mathbb{Z}\}$
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Note: $[0] = [-3] = [+3] = [-6] = [+6] = \cdots$ (from definition)

$[0] \neq [1] \neq [2]$ and $\mathbb{Z} = [0] \cup [1] \cup [2]$ (details in next slide)

Equivalence Classes and Partitions

Theorem: If ρ is an equivalence relation on $\mathcal{A}$ and $x, y \in \mathcal{A}$, then
(i) $x \in [x]$; **(ii)** $(x, y) \in \rho$ iff $[x] = [y]$; and **(iii)** $[x] = [y]$ or $[x] \cap [y] = \phi$
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Given set $\mathcal{A}$ and index set $\mathcal{I}$, let $\forall i,\ \phi \neq \mathcal{A}_i \subseteq \mathcal{A}$. Then $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ induces a partition on $\mathcal{A}$ if:
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Results: (i) Any equivalence relation ρ on set $\mathcal{A}$ induces a partition of $\mathcal{A}$.
Proof: Follows from the above theorem.

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(ii) Any partition of $\mathcal{A}$ gives rise to an equivalence relation ρ on $\mathcal{A}$.

Proof: Left for You as an Exercise!

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Partial Order: A relation $\rho \subseteq \mathcal{A} \times \mathcal{A}$ on set $\mathcal{A}$ is called a partial ordering relation (or partial order) if it is reflexive, antisymmetric and transitive.

We call $(\mathcal{A}, \rho)$ as a Poset (Partial Ordered Set).

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Example: Let $\mathcal{S} = \{1, 2, 3\}$ and $\rho = \{(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{S})\}$, therefore $(\mathcal{P}(\mathcal{S}), \rho)$ or $(\mathcal{P}(\mathcal{S}), \subseteq)$ is a poset.

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Covering Relation: Let $(\mathcal{A}, \rho)$ is a poset and $p, q, r \in \mathcal{A}$. We call q as the cover for p (denoted as $p \prec q$) when $(p, q) \in \rho$, and no element $r \in \mathcal{A}$ exists such that $p \prec r \prec q$, that is $(p, r) \in \rho$ and $(r, q) \in \rho$.

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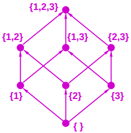

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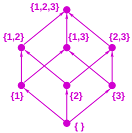

Total Order: If $(\mathcal{A}, \rho)$ is a Poset, we call $\mathcal{A}$ is totally ordered (or linearly ordered) if for all $x, y \in \mathcal{A}$ either $(x, y) \in \rho$ or $(y, x) \in \rho$. In this case, ρ is also called a total order (or linear order).

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If $(\mathcal{A}, \rho)$ is a poset has a least (greatest) element, then that element is unique.

Properties of Partial Orders

Lower Bound: Let $(\mathcal{A}, \rho)$ is a poset and $\mathcal{B} \subseteq \mathcal{A}$. An element $x \in \mathcal{A}$ is called a lower bound of $\mathcal{B}$ if $\forall b \in \mathcal{B},\ (x, b) \in \rho$.

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If $(\mathcal{A}, \rho)$ is a poset and $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B}$ has at most one lub (glb).

Lattice

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A lattice is a poset, $(\mathcal{A}, \rho)$, in which for every pair of elements $a, b \in \mathcal{A}$, the $lub\{a, b\}$ and $glb\{a, b\}$ both exists in $\mathcal{A}$.

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Example: *The following poset is NOT a lattice.*

Let $\mathcal{S} = \{1, 2, 3\}$ and $\mathcal{Q} \subset \mathcal{P}(\mathcal{S})$ (all proper subsets) where $\phi \notin \mathcal{Q}$. Poset $(\mathcal{Q}, \rho)$, where $\rho = \{(\mathcal{A}, \mathcal{B}) \mid \mathcal{A} \subseteq \mathcal{B} \text{ and } x, y \in \mathcal{Q}\}$ is NOT a lattice.
Here, the pair of elements $\{1, 2\}$ and $\{1, 3\}$ in $\mathcal{Q}$ do not have a *lub*, whereas the pair of elements $\{1\}$ and $\{2\}$ in $\mathcal{Q}$ do not have a *glb*.

Thank You!