# Functions

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# Basics of Functions

Functions: For two sets, $\mathcal{A}, \mathcal{B} \neq \phi$, a function (or mapping) $f$ from $\mathcal{A}$ to $\mathcal{B}$, denoted as $f : \mathcal{A} \rightarrow \mathcal{B}$, is a relation from $\mathcal{A}$ to $\mathcal{B}$ in which every element of $\mathcal{A}$ appears exactly once in the first component of an ordered pair in the relation.

$f(a) = b$ $(a \in \mathcal{A}, b \in \mathcal{B})$ when $(a, b)$ is an ordered pair in the function $f$ associating each $a$ to an unique $b$. Thus, $(a, b), (a, c) \in f \Rightarrow b = c$.

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**(2)** Floor and ceiling functions, $f : \mathbb{R} \rightarrow \mathbb{Z}$, are defined by,

$$f(x) = \lfloor x \rfloor \text{ and } g(y) = \lceil y \rceil \ (x, y \in \mathbb{R}).$$

$$f(2.7) = 2, f(-2.7) = -3, f(2) = 2, f(-2) = -2 \text{ and}$$

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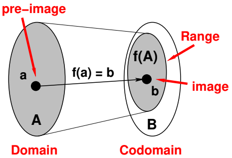

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Number of Functions: Let $\mathcal{A} = \{a_1, \ldots, a_m\}$ ($|\mathcal{A}| = m$) and $\mathcal{B} = \{b_1, \ldots, b_n\}$ ($|\mathcal{B}| = n$). $f : \mathcal{A} \rightarrow \mathcal{B}$ is described as, $\{(a_1, x_1), (a_2, x_2), \ldots, (a_m, x_m)\}$.

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**Extension:** Let $\mathcal{A}' \subseteq \mathcal{A}$ and $f : \mathcal{A}' \rightarrow \mathcal{B}$. If $g : \mathcal{A} \rightarrow \mathcal{B}$ and $g(a) = f(a)$ for all $a \in \mathcal{A}'$, then $g$ is called an extension of $f$ to $\mathcal{A}$.

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**Proof:** **(ii)** For each $b \in \mathcal{B}$, $b \in f(\mathcal{A}_1 \cap \mathcal{A}_2) \Rightarrow b = f(a)$, for some $a \in (\mathcal{A}_1 \cap \mathcal{A}_2)$
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**(i) and (ii)** *Left for You as an Exercise!*

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Examples: **(i)** $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 2x + 1,\ \forall x \in \mathbb{R}$ is one-to-one; because for all $x_1, x_2 \in \mathbb{R}$, we have
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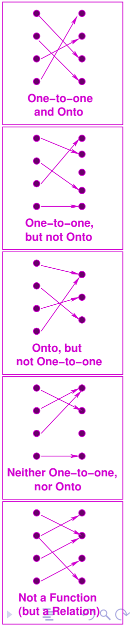

# (Binary) Operations and Properties

## Definition

Binary Operation: For non-empty sets, $\mathcal{A}, \mathcal{B}$, any function $f : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is called a binary operation on $\mathcal{A}$. If $\mathcal{B} \subseteq \mathcal{A}$ then the binary operation is closed on $\mathcal{A}$ (also $\mathcal{A}$ is closed under $f$). (Count: $|\mathcal{B}|^{|\mathcal{A}|^2}$)

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**Properties:** *Let $f : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$ is a binary operation.*

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**Projection:** For sets $\mathcal{A}, \mathcal{B}$, if $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{B}$, then –
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**Projection:** For sets $\mathcal{A}, \mathcal{B}$, if $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{B}$, then –
**(i)** $\pi_{\mathcal{A}} : \mathcal{C} \rightarrow \mathcal{A}$ defined by $\pi_{\mathcal{A}}(a,b) = a$, is called the projection on the first coordinate. **(ii)** $\pi_{\mathcal{B}} : \mathcal{C} \rightarrow \mathcal{B}$ defined by $\pi_{\mathcal{B}}(a,b) = b$, is called the projection on the second coordinate.

**Property:** If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, then $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ both are *onto* functions.

**Example:** Let $\mathcal{A} = \mathcal{B} = \mathbb{R}$ and $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{B}$ where $\mathcal{C} = \{(x,y) \mid y = x^2, \ x, y \in \mathbb{R}\}$ representing the Euclidean plane that contains points on the parabola $y = x^2$. Here, $\pi_{\mathcal{A}}(3,9) = 3$ and $\pi_{\mathcal{B}}(3,9) = 9$. Note that, $\pi_{\mathcal{A}}(\mathcal{C}) = \mathbb{R}$ and hence $\pi_{\mathcal{A}}$ is onto (and one-to-one as well). Whereas, $\pi_{\mathcal{B}}(\mathcal{C}) = [0, +\infty] \subset \mathbb{R}$ and hence $\pi_{\mathcal{B}}$ is NOT onto (nor it is one-to-one as $\pi_{\mathcal{B}}(2,4) = 4 = \pi_{\mathcal{B}}(-2,4)$).

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**Commutativity of Function Compositions:** *Does NOT Hold!*

Function Composition is NOT Commutative, that is, we shall NOT always have $f \circ g(x) \neq g \circ f(x)$ for any two functions, $f, g : \mathcal{A} \to \mathcal{A}$ (and $x \in \mathcal{A}$).

# Composite Function Properties

## Associativity of Function Compositions

If $f : \mathcal{A} \rightarrow \mathcal{B}$, $g : \mathcal{B} \rightarrow \mathcal{C}$ and $h : \mathcal{C} \rightarrow \mathcal{D}$, then $(h \circ g) \circ f = h \circ (g \circ f)$.

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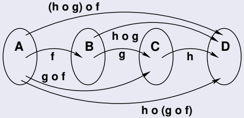

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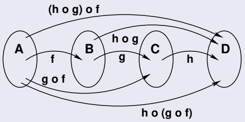


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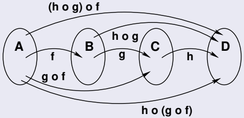


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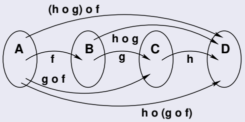


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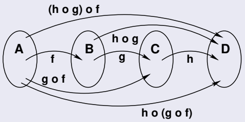


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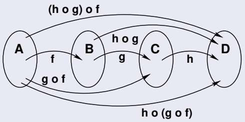


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**Proof:** For any $z \in \mathcal{C}$, $\exists y \in \mathcal{B}$ (as $g$ is onto) and $y \in \mathcal{B}$, $\exists x \in \mathcal{A}$ (as $f$ is onto).

So, $z = g(y) = g(f(x)) = (g \circ f)(x)$ and Range of $(g \circ f) = \mathcal{C} =$ Codomain of $(g \circ f)$.

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$g$ is onto (Proof): As $(g \circ f)$ is onto, for any $z \in \mathcal{C}$, $\exists x \in \mathcal{A}$ such that, $z = g \circ f(x) = g(f(x))$, implying that $z$ has a pre-image defined as $f(x) \in \mathcal{B}$ – thus making $g$ onto.

# Composite Function Properties

## Bijective Nature of Function Compositions

Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ and the composition $g \circ f : \mathcal{A} \rightarrow \mathcal{C}$ is a one-to-one (injective) function. Then, $f$ is one-to-one (however, $g$ need NOT be one-to-one).

**Explanation:**

$f$ is one-to-one (Proof): Assuming $f$ is NOT one-to-one, implies $\exists x_1, x_2 \in \mathcal{A}$ such that $f(x_1) = f(x_2)$. So, $g \circ f(x_1) = g \circ f(x_2)$, contradicting $g \circ f$ is injective!

$g$ is not one-to-one (Example): $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are defined as, $f(x) = e^x$ and $g(x) = x^2$ $(x \in \mathbb{R})$. Here, $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as, $g \circ f(x) = e^{2x}$. So, $(g \circ f)$ is one-to-one, but $g$ is NOT (note that, $f$ is one-to-one as proven)!

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Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ and the composition $g \circ f : \mathcal{A} \rightarrow \mathcal{C}$ is a onto (surjective) function. Then, $g$ is onto (however, $f$ need NOT be onto).

**Explanation:**

$g$ is onto (Proof): As $(g \circ f)$ is onto, for any $z \in \mathcal{C}$, $\exists x \in \mathcal{A}$ such that, $z = g \circ f(x) = g(f(x))$, implying that $z$ has a pre-image defined as $f(x) \in \mathcal{B}$ – thus making $g$ onto.

$f$ is not onto (Example): $f, g : \mathbb{Z} \rightarrow \mathbb{Z}$ are defined as, $f(x) = 2x$ and $g(x) = \lfloor \frac{x}{2} \rfloor$ $(x \in \mathbb{Z})$. Here, $g \circ f : \mathbb{Z} \rightarrow \mathbb{Z}$ is defined as, $g \circ f(x) = x$. So, $(g \circ f)$ is onto, but $f$ is NOT (note that, $g$ is onto as proven)!

# Inverse Functions and Invertibility

Inverse Functions: For a function $f : \mathcal{A} \to \mathcal{B}$, if $f_L^{-1}, f_R^{-1} : \mathcal{B} \to \mathcal{A}$ are defined such that $f_L^{-1} \circ f = 1_{\mathcal{A}}$ and $f \circ f_R^{-1} = 1_{\mathcal{B}}$, then $f_L^{-1}$ and $f_R^{-1}$ are called the left inverse and right inverse of $f$, respectively.

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**(2)** Let $f, g : \mathbb{R} \to \mathbb{R}$ are defined as $f(x) = 2x$ and $g(x) = \frac{x}{2}$ $(x \in \mathbb{R})$. So, $g \circ f, f \circ g : \mathbb{R} \to \mathbb{R}$ are defined by, $g \circ f(x) = g(2x) = x$ and $f \circ g(x) = f(\frac{x}{2}) = x$. So, $g \circ f = f \circ g = 1_{\mathbb{R}}$ meaning $g$ is inverse of $f$.

# Properties of Invertible Functions

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  So, $(g \circ f)$ is also bijective and hence invertible (using above property).

# Properties of Invertible Functions

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# Properties of Invertible Functions

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## Example

$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x + 1$ ($x \in \mathbb{R}$). Note that, $f$ is bijective (Why?) and hence invertible. Now, $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = \frac{y-1}{3}$, $y \in \mathbb{R}$.

# Properties with Direct and Inverse Images

Direct Image: Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and (non-empty) $\mathcal{A}' \subseteq \mathcal{A}$. The direct image of $\mathcal{A}'$ under $f$ is $f(\mathcal{A}') \subseteq \mathcal{B}$ given by, $f(\mathcal{A}') = \{f(x) \mid x \in \mathcal{A}'\}$.

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**Example:** $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ $(x \in \mathbb{R})$. Let $\mathcal{P} = \{x \in \mathbb{R} \mid x \in [0, 2]\}$. The direct image $f(\mathcal{P}) = \{y \mid y \in [0, 4]\}$ $(y \in \mathbb{R})$ and the inverse image of set $f(\mathcal{P})$ is $f^{-1}(f(\mathcal{P})) = \{x \mid x \in [-2, 2]\}$. So, $f^{-1}(f(\mathcal{P})) \neq \mathcal{P}$ and $f$ is not a bijection / invertible.

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**Properties:**

- (RECAP) Let $f : \mathcal{A} \to \mathcal{B}$, with $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$. Then, **(i)** If $\mathcal{A}_1 \subset \mathcal{A}_2 \Rightarrow f(\mathcal{A}_1) \subset f(\mathcal{A}_2)$, **(ii)** $f(\mathcal{A}_1 \cup \mathcal{A}_2) = f(\mathcal{A}_1) \cup f(\mathcal{A}_2)$, and **(iii)** $f(\mathcal{A}_1 \cap \mathcal{A}_2) \subset f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$.

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  **Note:** In general, $f(\mathcal{A}_1 \cap \mathcal{A}_2) \neq f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$. Consider, $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x^2$ and $\mathcal{A}_1 = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $\mathcal{A}_2 = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$. Here, $f(\mathcal{A}_1 \cap \mathcal{A}_2) = \{0\} \neq \{0, 1, \frac{1}{2^2}, \frac{1}{3^2}\} = f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$.

# Properties with Direct and Inverse Images

**Direct Image:** Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and (non-empty) $\mathcal{A}' \subseteq \mathcal{A}$. The direct image of $\mathcal{A}'$ under $f$ is $f(\mathcal{A}') \subseteq \mathcal{B}$ given by, $f(\mathcal{A}') = \{f(x) \mid x \in \mathcal{A}'\}$.

**Inverse Image:** Let $f : \mathcal{A} \rightarrow \mathcal{B}$ and (non-empty) $\mathcal{B}' \subseteq \mathcal{B}$. The inverse image (pre-image) of $\mathcal{B}'$ under $f$ is $f^{-1}(\mathcal{B}') \subseteq \mathcal{A}$ given by, $f^{-1}(\mathcal{B}') = \{x \mid f(x) \in \mathcal{B}'\}$.

**Example:** $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = x^2$ $(x \in \mathbb{R})$. Let $\mathcal{P} = \{x \in \mathbb{R} \mid x \in [0, 2]\}$. The direct image $f(\mathcal{P}) = \{y \mid y \in [0, 4]\}$ $(y \in \mathbb{R})$ and the inverse image of set $f(\mathcal{P})$ is $f^{-1}(f(\mathcal{P})) = \{x \mid x \in [-2, 2]\}$. So, $f^{-1}(f(\mathcal{P})) \neq \mathcal{P}$ and $f$ is not a bijection / invertible.

**Properties:**

- (RECAP) Let $f : \mathcal{A} \rightarrow \mathcal{B}$, with $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$. Then, **(i)** If $\mathcal{A}_1 \subset \mathcal{A}_2 \Rightarrow f(\mathcal{A}_1) \subset f(\mathcal{A}_2)$, **(ii)** $f(\mathcal{A}_1 \cup \mathcal{A}_2) = f(\mathcal{A}_1) \cup f(\mathcal{A}_2)$, and **(iii)** $f(\mathcal{A}_1 \cap \mathcal{A}_2) \subset f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$.

  **Note:** In general, $f(\mathcal{A}_1 \cap \mathcal{A}_2) \neq f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$. Consider, $f : \mathbb{R} \rightarrow \mathbb{R}$ as $f(x) = x^2$ and $\mathcal{A}_1 = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $\mathcal{A}_2 = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$. Here, $f(\mathcal{A}_1 \cap \mathcal{A}_2) = \{0\} \neq \{0, 1, \frac{1}{2^2}, \frac{1}{3^2}\} = f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$.

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- Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be an onto mapping, with $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$. Then, **(i)** If $\mathcal{B}_1 \subset \mathcal{B}_2 \Rightarrow f^{-1}(\mathcal{B}_1) \subset f^{-1}(\mathcal{B}_2)$, **(ii)** $\overline{f^{-1}(\mathcal{B}_1)} = f^{-1}(\overline{\mathcal{B}_1})$, **(iii)** $f^{-1}(\mathcal{B}_1 \cup \mathcal{B}_2) = f^{-1}(\mathcal{B}_1) \cup f^{-1}(\mathcal{B}_2)$, and **(iv)** $f^{-1}(\mathcal{B}_1 \cap \mathcal{B}_2) = f^{-1}(\mathcal{B}_1) \cap f^{-1}(\mathcal{B}_2)$.

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  **(ii), (iii) and (iv)** *Left for You as an Exercise!*

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If $\mathcal{A} = \{w, x, y, z\}$, $\mathcal{B} = \{1, 2, 3\}$, then all possible functions $= 3^4$;

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If $\mathcal{A} = \{w, x, y, z\}$, $\mathcal{B} = \{1, 2, 3\}$, then all possible functions $= 3^4$; this includes $2^4$ non-onto functions each from $\mathcal{A} \to \{1, 2\}$, $\mathcal{A} \to \{1, 3\}$ and $\mathcal{A} \to \{2, 3\}$. Now, the running count for onto functions $= 3^4 - 3.2^4$. But, we removed the constant function $\{(w, 2), (x, 2), (y, 2), (z, 2)\}$ twice – both during function removal from $\mathcal{A} \to \{1, 2\}$, $\mathcal{A} \to \{2, 3\}$.

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*What do the above steps reveal?* $\Rightarrow$ Principle of Inclusion-Exclusion!

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If $\mathcal{A} = \{w, x, y, z\}$, $\mathcal{B} = \{1, 2, 3\}$, then all possible functions $= 3^4$; this includes $2^4$ non-onto functions each from $\mathcal{A} \rightarrow \{1,2\}$, $\mathcal{A} \rightarrow \{1,3\}$ and $\mathcal{A} \rightarrow \{2,3\}$. Now, the running count for onto functions $= 3^4 - 3.2^4$. But, we removed the constant function $\{(w,2),(x,2),(y,2),(z,2)\}$ twice – both during function removal from $\mathcal{A} \rightarrow \{1,2\}$, $\mathcal{A} \rightarrow \{2,3\}$. So, the final onto functions count $= 3^4 - 3.2^4 + 3 = \binom{3}{3}3^4 - \binom{3}{2}2^4 + \binom{3}{1}1^4$.

If $|\mathcal{A}| = m \geq n = |\mathcal{B}|$, how many Onto functions? $= O(m, n)$

*What do the above steps reveal?* $\Rightarrow$ Principle of Inclusion-Exclusion!

$$O(m,n) = \binom{n}{n}n^m - \binom{n}{n-1}(n-1)^m + \binom{n}{n-2}(n-2)^m - \cdots + (-1)^{n-2}\binom{n}{2}2^m + (-1)^{n-1}\binom{n}{1}1^m$$

$$= \sum_{i=0}^{n-1}(-1)^i\binom{n}{n-i}(n-i)^m = \sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m$$

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## Combinatorial Definition

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Corollary: $\frac{1}{n}[n!.S(m+1,n)] = [(n-1)!.S(m,n-1)] + [n!.S(m,n)]$ (multiply by $(n-1)!$)

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$\Rightarrow \frac{1}{n}.O(m+1,n) = O(m,n-1) + O(m,n)$

## Counting Problems: *Are these problems well-recognized now?*

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# Thank You!