Uncountable = not countable

$$\left.\begin{array}{ll} f: A \to \mathbb{N} & \text{inj} \\ g: \mathbb{N} \to A & \text{surj} \end{array}\right\} A \text{ is countable.}$$

A is <u>not</u> countable

$\Leftrightarrow$ We cannot have an injective map $A \to \mathbb{N}$

$\Leftrightarrow$ No map $\mathbb{N} \to A$ can be bijective.

Diagonalization proof

Cantor

Theorem : $\mathbb{R}$ (the set of real numbers) is uncountable.

Proof : $[0,1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ is uncountable.

proper fractions

$$0.a_1 a_2 a_3 a_4 \ldots$$

each $a_i \in \{0, 1, 2, \ldots, 9\}$

$$\pi - 3 = 0.1415926535\ldots$$

$$3/8 = 0.375000\ldots$$
$$= 0.3749999\ldots$$

Let $f: \mathbb{N} \to [0,1)$ be any (injective) function.

We prove that $f$ cannot be surjective.

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

cannot be an exhaustive listing of the elements of $[0,1)$

No matter how we plan to enumerate the elements of $[0,1)$, we are bound to miss at least one element of $[0,1)$.

$$
\begin{array}{l}
f(1) = 0.\underline{a_{1,1}}\, a_{1,2}\, a_{1,3}\, a_{1,4}\, a_{1,5} \ldots a_{1,n} \ldots \\
f(2) = 0.a_{2,1}\, \underline{a_{2,2}}\, a_{2,3}\, a_{2,4}\, a_{2,5} \ldots a_{2,n} \ldots \\
f(3) = 0.a_{3,1}\, a_{3,2}\, \underline{a_{3,3}}\, a_{3,4}\, a_{3,5} \ldots a_{3,n} \ldots \\
f(4) = 0.a_{4,1}\, a_{4,2}\, a_{4,3}\, \underline{a_{4,4}}\, a_{4,5} \ldots a_{4,n} \ldots \\
f(5) = 0.a_{5,1}\, a_{5,2}\, a_{5,3}\, a_{5,4}\, \underline{a_{5,5}} \ldots a_{5,n} \ldots \\
\ldots \\
f(n) = 0.a_{n,1}\, a_{n,2}\, a_{n,3}\, a_{n,4}\, a_{n,5} \ldots \underline{a_{n,n}} \ldots \\
\ldots \\
\hline
\qquad b = 0.\ b_1\ \ b_2\ \ b_3\ \ b_4\ \ b_5\ \ldots\ b_n\ \ldots
\end{array}
$$

$$b_n = \begin{cases} 1 & \text{if } a_{n,n} = 2 \\ 2 & \text{if } a_{n,n} \neq 2 \end{cases}$$

$b_n$ is non-terminating

↳ has a unique decimal representation

$b_n \neq f(1), f(2), f(3), \ldots, f(n), \ldots$

$b \notin \mathrm{Im}(f) = f(\mathbb{N})$

Theorem: Let $\Sigma$ be an alphabet of size $|\Sigma| \geq 2$. Then the set of infinite sequences over $\Sigma$ is uncountable.

Proof: Same diagonalization proof works.

$k \geq 2$ (integer)

$k^{\aleph_0} > \aleph_0$

**Theorem**: There cannot be a bijection between a set $A$ and its power set $\wp(A) = 2^A$.

**Proof**: Let $f : A \to 2^A$ be an bijective function.

$$2^A = \{ f(a) \mid a \in A \}$$

$$B = \{ a \in A \mid a \notin f(a) \}$$

$f$ is bijective, so there exists $a \in A$ s.t. $B = f(a)$.

$$a \in f(a) \Rightarrow a \in B \Rightarrow a \notin f(a)$$

$$a \notin f(a) \Rightarrow a \notin B \Rightarrow a \in f(a)$$

$$a \in f(a) \Longleftrightarrow a \notin f(a).$$

$$f : A \to 2^A$$
$$a \longmapsto \{a\}$$

is injective.

$$|A| \leq |2^A|$$

No $f : A \to 2^A$ can be bijective

$$\Rightarrow |A| < |2^A|$$

---

$B \subseteq A$ $\quad$ $C_B : A \to \{0, 1\}$

$$a \longmapsto \begin{cases} 0 & \text{if } a \notin B \\ 1 & \text{if } a \in B \end{cases}$$

| $n$ | $f(n)$ | $C_{f(n)}(i)$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $\cdots$ |
| 1 | $\emptyset$ | $\underline{0}$ | 0 | 0 | 0 | 0 | $\cdots$ |
| 2 | $\{2,4,6,8\}$ | 0 | $\underline{1}$ | 0 | 1 | 0 | $\cdots$ |
| 3 | $\{2,3,5,7,11,\ldots\}$ | 0 | 1 | $\underline{1}$ | 0 | 1 | $\cdots$ |
| 4 | $\{1,3,5,7,9,\ldots\}$ | 1 | 0 | 1 | $\underline{0}$ | 1 | $\cdots$ |
| 5 | $\{1,2,3,5,8,13,\ldots\}$ | 1 | 1 | 1 | 0 | $\underline{1}$ | $\cdots$ |
| $\cdots$ | | | | | | | $\cdots$ |
| $B$ | $\{1,4,\ldots\}$ | 1 | 0 | 0 | 1 | 0 | $\cdots$ |

$A = \mathbb{N}$

$|\mathbb{R}| = \mathfrak{c}$ (continuum)

$\mathfrak{c} \geq |[0,1)| > \aleph_0$

$\mathfrak{c} > \aleph_0$

$f : \mathbb{R}_{\geq 0} \to [0,1)$

$x \longmapsto \dfrac{x}{x+1}$ is a bijection (easy exercise)

$|[0,1)| = |\mathbb{R}_{\geq 0}|$

$\mathbb{R}_{\geq 0} \subseteq \mathbb{R}$ $\quad \iota$ is injective

$|\mathbb{R}_{\geq 0}| \leq |\mathbb{R}|$

$g: \mathbb{R} \to \mathbb{R}_{\geq 0}$

$$x \mapsto \begin{cases} \dfrac{x}{x+1} & \text{if } x \geq 0 \\ \dfrac{-x}{-x+1} + 1 & \text{if } x < 0 \end{cases}$$

is injective

$|\mathbb{R}| = |\mathbb{R}_{\geq 0}| = |[0,1)| = c$

$2^{\aleph_0} = c$

$[0, 1)$ binary expansions

$0 \cdot 1101001 \ldots$

$0 \cdot 10000 \ldots = 0 \cdot 0111 \ldots$

Only some rational numbers have terminating expansion.

$\downarrow$

countable

There exist infinities larger than $c$.

$|2^{\mathbb{R}}| = 2^c > c \qquad 2^{2^c} > 2^c > c$

$\aleph_0$

$c = 2^{\aleph_0}$

$2^c$

$2^{2^c}$

There are in<u>finitely</u> many infinities

countable

Are there any other infinities?

Cantor: Is there an infinity strictly between $\aleph_0$ and $c$.

Continuum hypothesis:

No such infinity exists.

Could not be proved by Cantor.

Gödel - can be neither proved nor disproved

using the axioms of ZF set theory

Zermelo - Fraenkel

(even if you add the axiom of choice)