Sizes of Sets

Countable and uncountable sets

A - set

finite $|A|$ = the number of elements in A

$|A| < \infty$

Infinite sets $|A| = \infty$ (not clear)

Elements of a finite set can be counted.

$\mathbb{N} = \{1, 2, 3, 4, \ldots, n, \ldots\}$

Infinite counting process

There exist sets s.t. any inf counting fails to exhaust all the elements.

A, B two sets.

We say $|A| \leq |B|$ if there exists an injective map

$$f : A \to B$$

f produces an embedding of A in B.

Therefore B cannot be smaller than A.

Example (1) $A \subseteq B$

Inclusion map $\iota : A \to B$ is injective

$$a \mapsto a$$

$$|A| \leq |B|$$

(2) $|\mathbb{N}| \leq |\mathbb{Z}|$

$|\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{R}|$

$|\mathbb{N}_{odd}| \leq |\mathbb{N}|$

(3) $|\mathbb{Z}| \leq |\mathbb{N}|$

$0 \mapsto 1$

$1 \mapsto 2$

$-1 \mapsto 3$

$2 \mapsto 4$

$-2 \mapsto 5$

$\dots$

$n \mapsto 2n$

$-n \mapsto 2n+1$

$\dots$

bijective

$0 \mapsto 1$

$1 \mapsto 3$

$-1 \mapsto 5$

$2 \mapsto 7$

$-2 \mapsto 9$

$\dots$

not a bijection

Def: $|A| = |B|$ if

$|A| \leq |B|$ **and** $|B| \leq |A|$

or equivalently if there exist injective maps $f: A \to B$ and $g: B \to A$.

Example: $|\mathbb{N}| = |\mathbb{Z}|$

$|A| = |B| \Rightarrow$ A and B are equinumerous

Theorem [Cantor – Schröder – Bernstein]

$|A| = |B|$ if and only if there exists a bijective map $h: A \to B$.

Countable sets

Theorem: Let A be any infinite set.

Then $|\mathbb{N}| \leq |A|$.

Proof: $f : \mathbb{N} \to A$ injective.

$a_1 \in A$ $\qquad f(1) = a_1$

$f(i) = a_i$ for $i = 1, 2, 3, \ldots, n$

← distinct

A is infinite. So we can choose a_{n+1} from A different from $a_1, a_2, \ldots, a_n$, and define

$f(n+1) = a_{n+1}$. Induction.

Corollary: $|\mathbb{N}|$ is the smallest infinity.

$$|\mathbb{N}| = \aleph_0 \quad \text{(Aleph-not)}$$

Def: A set A is called countable if $|A| < \infty$ or $|A| = |\mathbb{N}|$.

A is countable (and infinite)

$\Leftrightarrow$ $\exists$ an injective map $f: A \to \mathbb{N}$

$\Leftrightarrow$ $\exists$ a bijective map $h: \mathbb{N} \to A$ [CSB theorem]

$A = \{ h(1), h(2), h(3), \ldots, h(n), \ldots \}$ infinite counting

Theorem: Any subset of a countable set is countable.

Proof: $A \subseteq B$ (B countable)

$|A| < \infty$, we are done.

$\iota : A \to B$ (inclusion map)

injective

$$\left.\begin{array}{l} |A| \leq |B| = |\mathbb{N}| \\ |\mathbb{N}| \leq |A| \end{array}\right\} \quad |A| = |\mathbb{N}|$$

Theorem: The union of two countable sets A, B is again countable.

Proof: $A = \{a_1, a_2, a_3, a_4, \ldots\}$

$B = \{b_1, b_2, b_3, b_4, \ldots\}$

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots\}$$

if $A \cap B \neq \emptyset$, then do not list the second appearances.

Theorem: Let $k \in \mathbb{N}$, and

$A_1, A_2, \ldots, A_k$ countable sets.

Then $\bigcup_{i=1}^{k} A_i$ is countable.

Proof: $k = 1$ ✓

$k \geq 1$ $\quad B = \bigcup_{i=1}^{k} A_i$ is countable.

$\bigcup_{i=1}^{k+1} A_i = B \cup A_{k+1}$. Use previous theorem.

__Theorem__: The union of countably many countable sets is again countable.

__Proof__: $A_n, n \in \mathbb{N}$, a collection of countable sets.

$$A_n = \{a_{n,1}, a_{n,2}, a_{n,3}, \dots, a_{n,m}, \dots\}$$

$$a_{i,j} \qquad \begin{array}{l} i = 1, 2, 3, \dots \\ j = 1, 2, 3, \dots \end{array}$$

$$
\begin{matrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & \cdots \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & \cdots \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & \cdots \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & \cdots \\
a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{matrix}
$$

Exhaustive enumeration of all the elements $a_{i,j}$.

Don't include the same elements multiple times.

__Corollary__ : A, B countable

Then $A \times B$ is again countable.

__Proof__ : $\forall a \in A$, define

$$B_a = \{(a, b) \mid b \in B\}$$

$$\begin{aligned} B &\longmapsto B_a \\ b &\longmapsto (a, b) \end{aligned}$$ bijection

Each B_a is countable.

$$A \times B = \bigcup_{a \in A} B_a.$$

Corollary: $\mathbb{Q}$ is countable.

Proof: $\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{N},\ \gcd(a,b) = 1 \right\}$

$$\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{N}.$$

$\mathbb{Q}$ $\downarrow$ countable

$\mathbb{Z} \times \mathbb{N}$ $\downarrow$ countable

$\aleph_0 + \aleph_0 = \aleph_0$

$k \in \mathbb{N}, \quad k\aleph_0 = \aleph_0$

$\aleph_0 \times \aleph_0 = \aleph_0$

$k \in \mathbb{N} \quad \aleph_0^k = \aleph_0$

$|\mathbb{N}_{odd}| = |\mathbb{N}_{even}| = |\mathbb{N}| = |\mathbb{Z}_{odd}|$
$= |\mathbb{Z}_{even}| = |\mathbb{Z}| = |\mathbb{P}| = |\mathbb{Q}| = \aleph_0$

$\mathbb{R}$ is <u>not countable</u>.