Pigeon-Hole Principle

Dirichlet's box principle

Theorem: If m pigeons occupy n holes and $m > n$, then there is a hole containing more than one pigeons.

Theorem: If m pigeons occupy n holes, then there is a hole that contains at least $\lceil m/n \rceil$ pigeons.

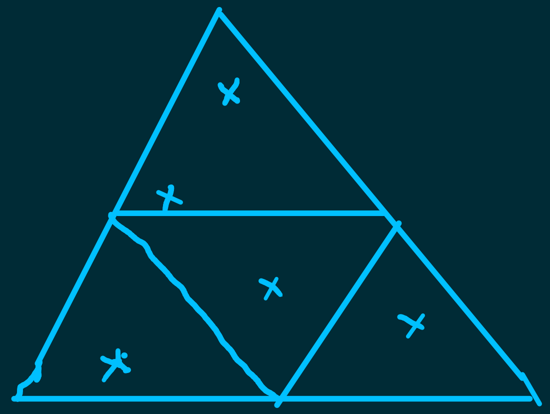
Proof

$$n\left(\lceil m/n \rceil - 1\right) < n\left(\frac{m}{n} + 1 - 1\right)$$

$$= m$$

Example 1

Equilateral triangle



Each side is of length 1.

Five points inside the triangle.

Two of these points at distance no more than 1/2

Example 2

$S = \{1, 2, 3, \ldots, 2n\}$

Pick $n+1$ numbers from S.

Then you must have picked up two numbers a, b such that $a \mid b$.

$\{1, 2, 2^2, \ldots\}$

$\{3, 3\times 2, 3\times 2^2, \ldots\}$

$\{5, 5\times 2, 5\times 2^2, \ldots\}$

$\{7, 7\times 2, 7\times 2^2, \ldots\}$

$\vdots$

$\{2n-1\}$

Example 3

n positive integers

$a_1, a_2, \dots, a_n$

There exists a non-empty subcollection of these numbers, whose sum is divisible by n.

$p_0 = 0$ $\qquad r_0$

$p_1 = a_1$ $\qquad r_1$ $\qquad r_i = p_i \text{ rem } n$

$p_2 = a_1 + a_2$ $\qquad r_2$

$p_3 = a_1 + a_2 + a_3$ $\qquad r_i = r_j$

$\vdots$ $\qquad$ for some $i < j$

$\dots$

$p_n = a_1 + a_2 + \dots + a_n$ $\qquad r_n$

$$p_i = q_i n + r_i$$

$$p_j = q_j n + r_j$$

$$p_j - p_i = (q_i - q_j) n$$

$$= a_{i+1} + a_{i+2} + \cdots + a_j$$

Example 4

Let m be an odd positive integer. There exists a positive integer n such that m divides $2^n - 1$.

Proof: $2^1, 2^2, 2^3, \ldots, 2^{m+1}$

$r_1, r_2, r_3, \ldots, r_{m+1}$ (remainder of division by m)

$r_i = r_j$ for $i < j$

$$2^i = q_i m + r_i$$

$$2^j = q_j m + r_j$$

$$2^j - 2^i = (q_i - q_j) m$$

$$= 2^i (2^{j-i} - 1)$$

m is odd

$\gcd(m, 2^i) = 1$

$$m \mid (2^{j-i} - 1)$$

Example 5

40 assignments

28 days

On each day, you solve at least one assignment

There must exist a set of consecutive days on which you solve 15 assignments.

Proof: x_i = the no of assignments solved from Day 1 to Day i

$$1 \leq x_1 < x_2 < x_3 < \cdots < x_{28} = 40$$

$$y_i = x_i + 15$$

$$16 \leq y_1 < y_2 < y_3 < \cdots < y_{28} = 55$$

$$56 \text{ numbers} \quad \left.\begin{array}{l} x_1, \dots, x_{28} \\ y_1, \dots, y_{28} \end{array}\right\}$$

$$x_i = y_j = x_j + 15$$

$$x_i - x_j = 15$$

Example 6

A of n^2+1 distinct numbers.

Then A contains either an increasing or a decreasing subsequence of length (at least) $n+1$.

<u>Proof</u>: x_i = the length of the longest increasing subsequence ending at the ith array element

y_i = the length of a longest decreasing subsequence ending at the ith array element.

$(x_1, y_1), (x_2, y_2), \dots, (x_{n^2+1}, y_{n^2+1})$

n^2+1 such pairs

If no inc/dec subsequence of length $n+1$ (or more), there are at most n^2 values of (x_i, y_i)

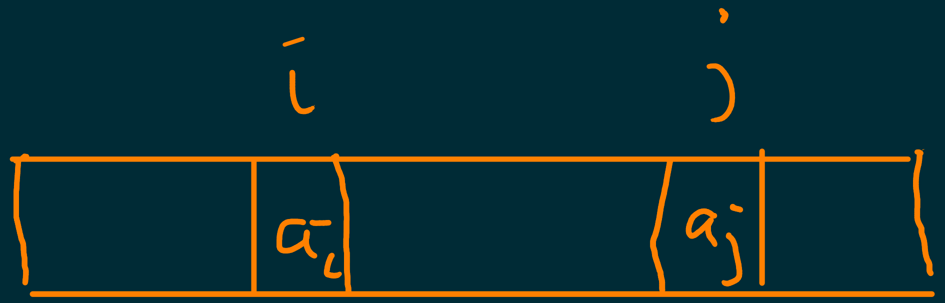
a_i is the ith array element

$$(x_i, y_i) = (x_j, y_j)$$

$$x_i = x_j$$

$$y_i = y_j$$

$$j > i$$



$x_i = x_j$

$y_i = y_j$

$a_i < a_j$

$a_i > a_j$

Chinese remainder theorem

(CRT)

$$m, n \in \mathbb{Z}^+, \quad \gcd(m, n) = 1$$

$$r \in \{0, 1, 2, \ldots, m-1\} = \mathbb{Z}_m$$

$$s \in \{0, 1, 2, \ldots, n-1\} = \mathbb{Z}_n$$

There exists a unique integer x in the range $0, 1, 2, \ldots, mn - 1$ such that

$$x \text{ rem } m = r \quad \text{and}$$

$$x \text{ rem } n = s$$

Proof: $\{0, 1, 2, \ldots, mn-1\}$

$r, r+m, r+2m, \ldots, r+(n-1)m$

$s_0, s_1, s_2, \ldots, s_{n-1}$

$s_i = (r+im) \text{ rem } n$

No s_i is equal to s (Assumption)

$n-1$ possible remainder values

n remainders $s_0, s_1, \ldots, s_{n-1}$

$s_i = s_j$ for $0 \le i < j \le n-1$.

$$r + im = q_i n + s_i$$

$$r + jm = q_j n + s_j$$

$$(j-i)m = (q_j - q_i)n$$

$$n \mid (j-i)m \qquad \gcd(m,n) = 1$$

$$n \mid j-i \qquad 0 \leq i < j \leq n-1$$

$$0 < j-i \leq n-1$$

Uniqueness

$x, y \in \{0, 1, 2, \ldots, mn-1\}$

$x \text{ rem } m = y \text{ rem } m = r$

$x \text{ rem } n = y \text{ rem } n = s$

$x = q_1 m + r$

$y = q_2 m + r$

$x = q_3 n + s$

$y = q_4 n + s$

$x - y = (q_1 - q_2) m = (q_3 - q_4) n$

$mn \mid x - y$

$y = x + kmn$